CSE 421: Introduction to Algorithms

Network Flow

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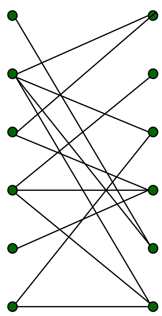
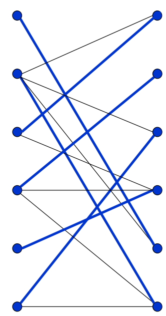
Bipartite Matching

- Given: A bipartite graph $G=(V,E)$
  - $M \subseteq E$ is a matching in G iff no two edges in M share a vertex
- Goal: Find a matching M in G of maximum possible size

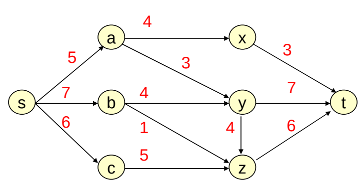
Bipartite Matching



Bipartite Matching

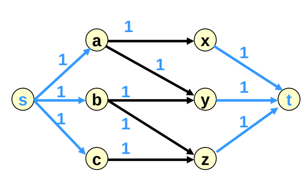


The Network Flow Problem



- How much stuff can flow from s to t?

Bipartite matching as a special case of flow



Net Flow: Formal Definition

Given:

A digraph **G = (V,E)**

Two vertices **s,t** in **V** (source & sink)

A *capacity* **c(u,v) ≥ 0** for each **(u,v) ∈ E** (and **c(u,v) = 0** for all non-edges **(u,v)**)

Find:

A *flow function* **f: E → R** s.t., for all **u,v**:

- **0 ≤ f(u,v) ≤ c(u,v)** [Capacity Constraint]
- if **u ≠ s,t**, i.e. $f^{out}(u)=f^{in}(u)$ [Flow Conservation]

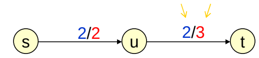
Maximizing total flow $v(f) = f^{out}(s)$

Notation:

$$f^{in}(v) = \sum_{e=(u,v)\in E} f(u,v) \qquad f^{out}(v) = \sum_{e=(v,w)\in E} f(v,w)$$

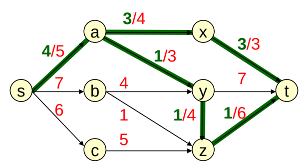
Example: A Flow Function

flow/capacity, not .66...



$f^{in}(u)=f(s,u)=2=f(u,t)=f^{out}(u)$

Example: A Flow Function

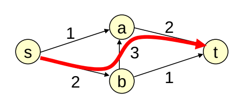


- Not shown: f(u,v) if = 0
- Note: max flow ≥ 4 since f is a flow function, with v(**f**) = 4

Max Flow via a Greedy Alg?

While there is an s → t path in G
  Pick such a path, **p**
  Find **c**, the min capacity of any edge in **p**
  Subtract **c** from all capacities on **p**
  Delete edges of capacity **0**

- This does NOT always find a max flow:



If pick s →b →a →t first, flow stuck at 2.
But flow 3 possible.

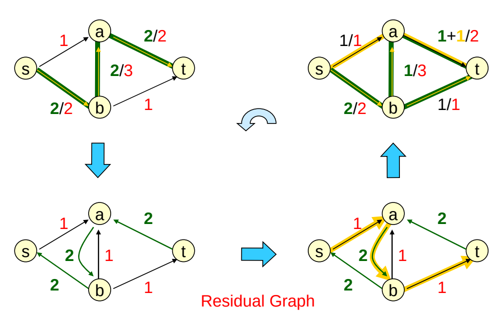
A Brief History of Flow

#	year	discoverer(s)	bound
1	1951	Dantzig	$O(n^2mU)$
2	1955	Ford & Fulkerson	$O(nmU)$
3	1970	Dinitz Edmonds & Karp	$O(nm^2)$
4	1970	Dinitz	$O(n^2m)$
5	1972	Edmonds & Karp Dinitz	$O(m^2 \log U)$
6	1973	Dinitz Gabow	$O(nm \log U)$
7	1974	Karzanov	$O(n^3)$
8	1977	Cherkassky	$O(n^2\sqrt{m})$
9	1980	Galil & Naamad	$O(nm \log^2 n)$
10	1983	Sleator & Tarjan	$O(nm \log n)$
11	1986	Goldberg & Tarjan	$O(nm \log(n^2/m))$
12	1987	Ahuja & Orlin	$O(nm + n^2 \log U)$
13	1987	Ahuja et al.	$O(nm \log(n\sqrt{\log U}/(m+2)))$
14	1989	Cheriyan & Hagerup	$E(nm + n^2 \log^2 n)$
15	1990	Cheriyan et al.	$O(n^3/\log n)$
16	1990	Alon	$O(nm + n^{8/3} \log n)$
17	1992	King et al.	$O(nm + n^{2+\epsilon})$
18	1993	Phillips & Westbrook	$O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$
19	1994	King et al.	$O(nm \log_{m/(n \log n)} n)$
20	1997	Goldberg & Rao	$O(m^{3/2} \log(n^2/m) \log U)$ $O(n^{2/3}m \log(n^2/m) \log U)$

n = # of vertices
m= # of edges
U = Max capacity

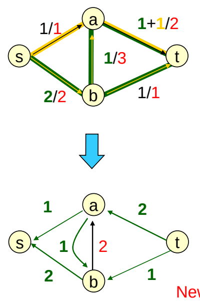
Source: Goldberg & Rao, FOCS '97

Greed Revisited: Residual Graph & Augmenting Path



Residual Graph

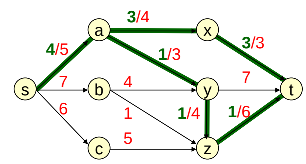
Greed Revisited: An Augmenting Path



New Residual Graph

Residual Capacity

- The *residual capacity* (w.r.t. **f**) of **(u,v)** is **c_f(u,v) = c(u,v) - f(u,v)** if **f(u,v)≤c(u,v)** and **c_f(u,v)=f(v,u)** if **f(v,u)**>0

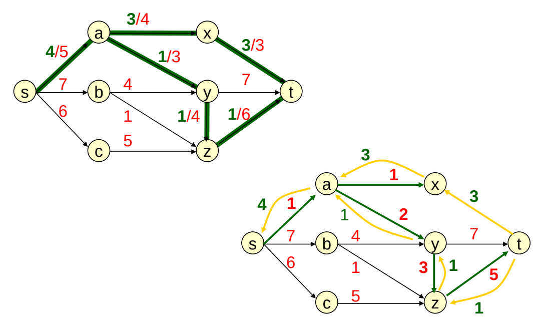


- e.g. **c_f(s,b)=7**; **c_f(a,x) = 1**; **c_f(x,a) = 3**

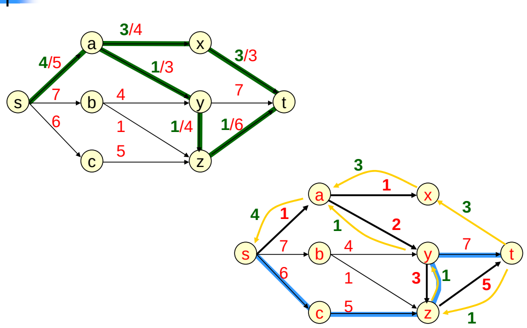
Residual Graph & Augmenting Paths

- The *residual graph* (w.r.t. **f**) is the graph **G_f = (V,E_f)**, where
  E_f = { (u,v) | c_f(u,v) > 0 }
  - Two kinds of edges
    - Forward edges
      - **f(u,v)<c(u,v)** so **c_f(u,v)=c(u,v)-f(u,v)>0**
    - Backward edges
      - **f(u,v)>0** so **c_f(v,u) ≥ -f(v,u)=f(u,v)>0**
- An *augmenting path* (w.r.t. **f**) is a simple **s → t** path in **G_f**.

A Residual Network



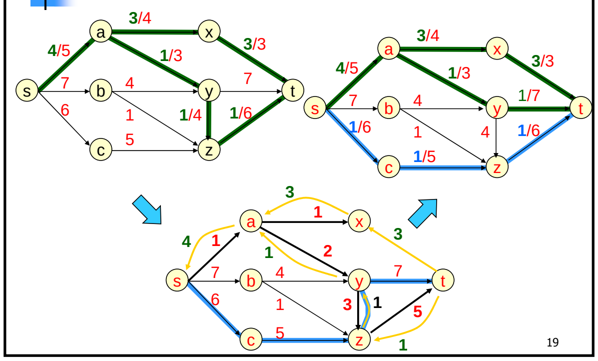
An Augmenting Path



Augmenting A Flow

augment(**f**,**P**)
  c_P←min$_{(u,v)\in P}$ **c_f(u,v)**   "bottleneck(P)"
  for each **e∈P**
    if **e** is a forward edge then
      increase **f(e)** by **c_P**
    else (**e** is a backward edge)
      decrease **f(e)** by **c_P**
    endif
  endfor
  return(**f**)

Augmenting A Flow



Claim 7.1

If G_f has an augmenting path **P**, then the function **f'=augment(f,P)** is a legal flow.

Proof:

- **f'** and **f** differ only on the edges of **P** so only need to consider such edges **(u,v)**

Proof of Claim 7.1

- If **(u,v)** is a forward edge then
  $f'(u,v)=f(u,v)+c_P \leq f(u,v)+c_f(u,v)$
  $= f(u,v)+c(u,v)-f(u,v)$
  $=c(u,v)$
- If **(u,v)** is a backward edge then **f** and **f'** differ on flow along **(v,u)** instead of **(u,v)**
  $f'(v,u)=f(v,u)-c_P \geq f(v,u)-c_f(u,v)$
  $= f(v,u)-f(v,u)=0$
- Other conditions like flow conservation still met

Ford-Fulkerson Method

Start with **f=0** for every edge

While G_f has an augmenting path,
augment

- Questions:
  - Does it halt?
  - Does it find a maximum flow?
  - How fast?

Observations about Ford-Fulkerson Algorithm

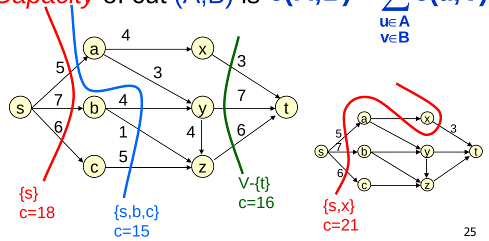
- At every stage the capacities and flow values are always integers (if they start that way)
- The flow value $v(f')=v(f)+c_P>v(f)$ for **f'=augment(f,P)**
  - Since edges of residual capacity **0** do not appear in the residual graph
- Let $C=\sum_{(s,u)\in E} c(s,u)$
  - $v(f)\leq C$
  - **F-F** does at most **C** rounds of augmentation since flows are integers and increase by at least **1** per step

Running Time of Ford-Fulkerson

- For **f=0**, $G_f=G$
- Finding an augmenting path in G_f is graph search $O(n+m)=O(m)$ time
- Augmenting and updating G_f is $O(n)$ time
- Total $O(mC)$ time
- Does is find a maximum flow?
  - Need to show that for every flow **f** that isn't maximum G_f contains an **s-t**-path

Cuts

- A partition **(A,B)** of **V** is an *s-t-cut* if
  - $s \in A$, $t \in B$
- *Capacity* of cut (A,B) is $c(A,B) = \sum_{u \in A, v \in B} c(u,v)$

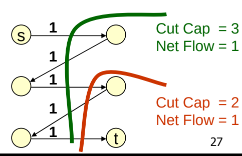


Convenient Definition

- $f^{out}(A) = \Sigma_{v \in A,\, w \notin A}\, f(v,w)$
- $f^{in}(A) = \Sigma_{v \in A,\, u \notin A}\, f(u,v)$

Claims 7.6 and 7.8

- For any flow **f** and any cut **(A,B)**,
  - the net flow across the cut equals the total flow, i.e., $v(f) = f^{out}(A) - f^{in}(A)$, and
  - the net flow across the cut cannot exceed the capacity of the cut, i.e. $f^{out}(A) - f^{in}(A) \le c(A,B)$
- Corollary :
  Max flow ≤ Min cut



Proof of Claim 7.6

- Consider a set **A** with $s \in A$, $t \notin A$
- $f^{out}(A) - f^{in}(A) = \Sigma_{v \in A,\, w \notin A}\, f(v,w) - \Sigma_{v \in A,\, u \notin A}\, f(u,v)$
- We can add flow values for edges with both endpoints in **A** to **both** sums and they would cancel out so
- $f^{out}(A) - f^{in}(A) = \Sigma_{v \in A,\, w \in V}\, f(v,w) - \Sigma_{v \in A,\, u \in V}\, f(u,v)$
  $= \Sigma_{v \in A}\, (\Sigma_{w \in V}\, f(v,w) - \Sigma_{u \in V}\, f(u,v))$
  $= \Sigma_{v \in A}\, f^{out}(v) - f^{in}(v)$
  $= f^{out}(s) - f^{in}(s)$

  since all other vertices have $f^{out}(v) = f^{in}(v)$
- $v(f) = f^{out}(s)$ and $f^{in}(s) = 0$

Proof of Claim 7.8

- $v(f) = f^{out}(A) - f^{in}(A)$
  $\le f^{out}(A)$
  $= \Sigma_{v \in A,\, w \notin A}\, f(v,w)$
  $\le \Sigma_{v \in A,\, w \notin A}\, c(v,w)$
  $\le \Sigma_{v \in A,\, w \in B}\, c(v,w)$
  $= c(A,B)$

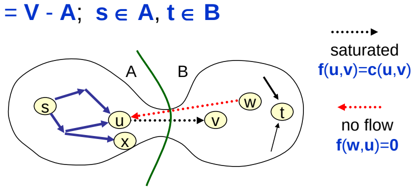
Max Flow / Min Cut Theorem

Claim 7.9 For any flow **f**, if G_f has no augmenting path then there is some **s-t**-cut **(A,B)** such that $v(f) = c(A,B)$ (proof on next slide)

- We know by Claims 7.6 & 7.8 that any flow **f'** satisfies $v(f') \le c(A,B)$ and we know that F-F runs for finite time until it finds a flow **f** satisfying conditions of Claim 7.9
  - Therefore by 7.9 for any flow **f'**, $v(f') \le v(f)$
- Corollary (1) F-F computes a maximum flow in **G**
  (2) For any graph **G**, the value $v(f)$ of a maximum flow = minimum capacity $c(A,B)$ of any **s-t**-cut in **G**

Claim 7.9

Let $A = \{ u \mid \exists$ an path in G_f from s to $u \}$
$B = V - A$; $s \in A$, $t \in B$



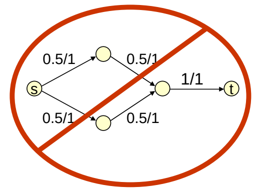
This is true for **every** edge crossing the cut, i.e.

$f^{out}(A) = \sum_{u\in A, v\in B} f(u,v) = \sum_{u\in A, v\in B} c(u,v) = c(A,B)$ and $f^{in}(A)=0$ so $v(f)=f^{out}(A)-f^{in}(A)=c(A,B)$

Flow Integrality Theorem

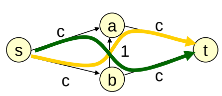
If all capacities are integers

- The max flow has an integer value
- Ford-Fulkerson method finds a max flow in which $f(u,v)$ is an integer for all edges (u,v)



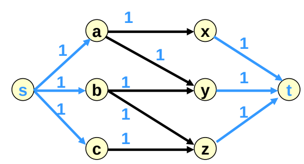
Corollaries & Facts

- If Ford-Fulkerson terminates, then it's found a max flow.
- It will terminate if $c(e)$ integer or rational (but may not if they're irrational).
- However, may take exponential time, even with integer capacities:



$c = 10^9$, say

Bipartite matching as a special case of flow



Integer flows implies each flow is just a subset of the edges

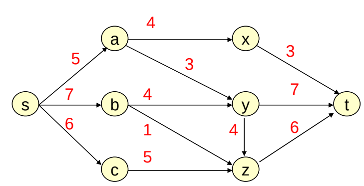
Therefore flow corresponds to a matching

$O(mC)=O(nm)$ running time

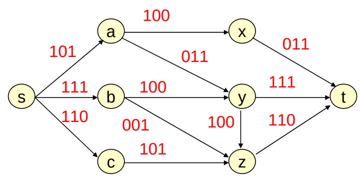
Capacity-scaling algorithm

- General idea:
  - Choose augmenting paths P with 'large' capacity c_P
  - Can augment flows along a path P by any amount $b \le c_P$
    - Ford-Fulkerson still works
  - Get a flow that is maximum for the high-order bits first and then add more bits later

Capacity Scaling



Capacity Scaling



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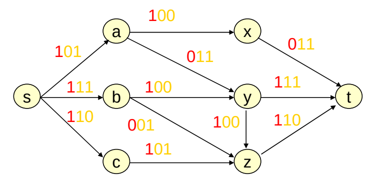
Capacity Scaling Bit 1



Capacity on each edge is at most **1**

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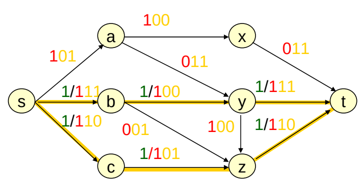
Capacity Scaling Bit 1



O(nm) time

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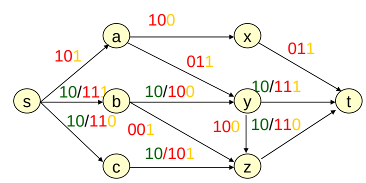
Capacity Scaling Bit 2



Residual capacity across min cut is at most **m**

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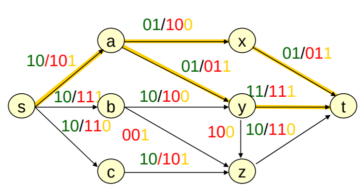
Capacity Scaling Bit 2



Residual capacity across min cut is at most **m**

⇒ **O(m)** augmentations

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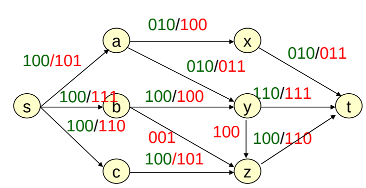
Capacity Scaling Bit 3



Residual capacity across min cut is at most **m**

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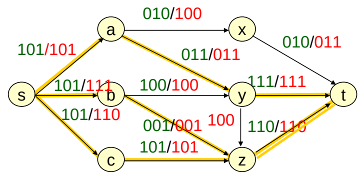
Capacity Scaling Bit 3



After O(**m**) augmentations

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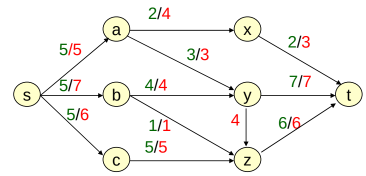
Capacity Scaling Final



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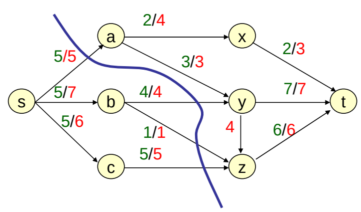
Capacity Scaling Min Cut



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Total time for capacity scaling

- $\log_2 \mathbf{U}$ rounds where **U** is largest capacity
- At most **m** augmentations per round
  - Let $\mathbf{c_i}$ be the capacities used in the $\mathbf{i}^{th}$ round and $\mathbf{f_i}$ be the maxflow found in the $\mathbf{i}^{th}$ round
    - For any edge $(\mathbf{u},\mathbf{v})$, $\mathbf{c_{i+1}}(\mathbf{u},\mathbf{v}) \leq \mathbf{2c_i}(\mathbf{u},\mathbf{v})\mathbf{+1}$
  - $\mathbf{i+1}^{st}$ round starts with flow $\mathbf{f} = \mathbf{2\ f_i}$
  - Let $(\mathbf{A},\mathbf{B})$ be a min cut from the $\mathbf{i}^{th}$ round
    - $\mathbf{v}(\mathbf{f_i})=\mathbf{c_i}(\mathbf{A},\mathbf{B})$ so $\mathbf{v}(\mathbf{f})=\mathbf{2c_i}(\mathbf{A},\mathbf{B})$
  - $\mathbf{v}(\mathbf{f_{i+1}}) \leq \mathbf{c_{i+1}}(\mathbf{A},\mathbf{B}) \leq \mathbf{2c_i}(\mathbf{A},\mathbf{B})\mathbf{+m} = \mathbf{v}(\mathbf{f})\mathbf{+m}$
- O(**m**) time per augmentation
- Total time O($\mathbf{m^2}$ log **U**)

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Edmonds-Karp Algorithm

- Use a shortest augmenting path
  (via Breadth First Search in residual graph)
- Time: O($\mathbf{n\ m^2}$)

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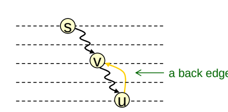
BFS/Shortest Path Lemmas

Distance from **s** in $\mathbf{G_f}$ is never reduced by:

- Deleting an edge
  Proof: no new (hence no shorter) path created
- Adding an edge (u,v), provided v is nearer than u
  Proof: BFS is unchanged, since **v** visited before **(u,v)** examined



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Key Lemma

Let f be a flow, G_f the residual graph, and P a shortest augmenting path. Then no vertex is closer to **s** after augmentation along **P**.

Proof: Augmentation along **P** only deletes forward edges, or adds back edges that go to previous vertices along **P**

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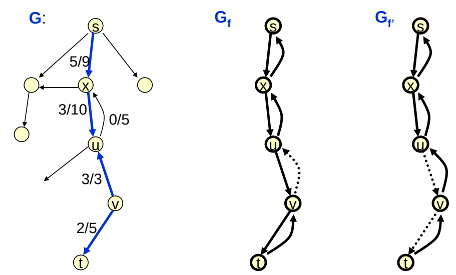
Augmentation vs BFS



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Theorem

The Edmonds-Karp Algorithm performs $O(mn)$ flow augmentations

Proof:

Call (u,v) critical for augmenting path **P** if it's closest to **s** having min residual capacity

It will disappear from G_f after augmenting along **P**

In order for **(u,v)** to be critical again the **(u,v)** edge must re-appear in G_f but that will only happen when the distance to **u** has increased by **2** (next slide)

It won't be critical again until farther from **s** so each edge critical at most **n/2** times

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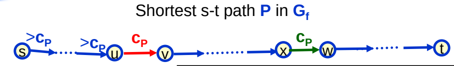
Critical Edges in G_f

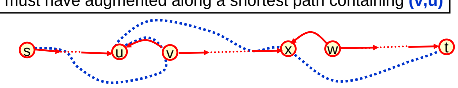
Shortest s-t path **P** in G_f



critical edge $d_f(s,v)=d_f(s,u)+1$ since this is a shortest path

After augmenting along **P**

For **(u,v)** to be critical later for some flow **f'** it must be in $G_{f'}$ so must have augmented along a shortest path containing **(v,u)**



Then we must have $d_{f'}(s,u)=d_{f'}(s,v)+1 \geq d_f(s,v)+1=d_f(s,u)+2$

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Corollary

- Edmonds-Karp runs in $O(nm^2)$ time

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Project Selection a.k.a. The Strip Mining Problem

- Given
  - a directed acyclic graph $G=(V,E)$ representing precedence constraints on tasks (a task points to its **predecessors**)
  - a profit value $p(v)$ associated with each task $v \in V$ (may be positive or negative)
- Find
  - a set $A \subseteq V$ of tasks that is closed under predecessors, i.e. if $(u,v) \in E$ and $u \in A$ then $v \in A$, that maximizes $\text{Profit}(A)=\sum_{v \in A} p(v)$

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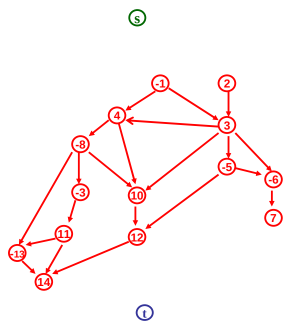
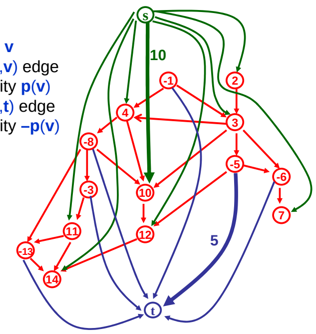
Extended Graph



Extended Graph G’

For each vertex **v**
If **p(v)≥0** add **(s,v)** edge
with capacity **p(v)**
If **p(v)<0** add **(v,t)** edge
with capacity **–p(v)**

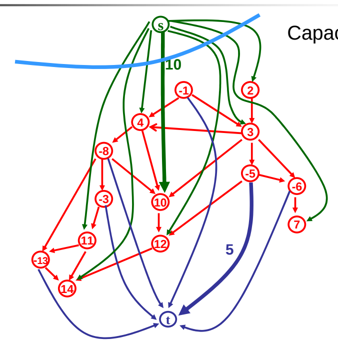


Extended Graph G’

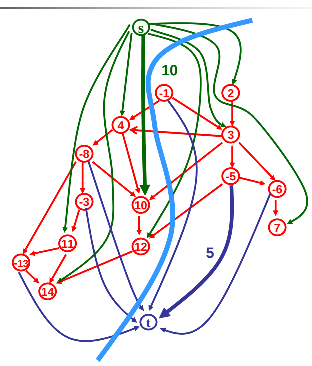
- Want to arrange capacities on edges of **G** so that for minimum **s**-**t**-cut **(S,T)** in **G’**, the set **A=S-{s}**
  - satisfies precedence constraints
  - has maximum possible profit in **G**
- Cut capacity with **S={s}** is just $C=\Sigma_{v:\ p(v)\geq 0}\ p(v)$
  - Profit(**A**) ≤ **C** for any set **A**
- To satisfy precedence constraints don’t want any original edges of **G** going forward across the minimum cut
  - That would correspond to a task in **A=S-{s}** that had a predecessor not in **A=S-{s}**
- Set capacity of each of these edges to **C+1**
  - The minimum cut has size at most **C**

Extended Graph G’

Capacity **C**



Extended Graph G’



Project Selection

- **Claim** Any **s**-**t**-cut **(S,T)** in **G’** such that **A=S-{s}** satisfies precedence constraints has capacity

  $c(S,T)=C - \Sigma_{v\in A}\ p(v) = C - \mathrm{Profit}(A)$

- **Corollary** A minimum cut **(S,T)** in **G’** yields an optimal solution **A=S-{s}** to the profit selection problem

- **Algorithm** Compute maximum flow **f** in **G’**, find the set **S** of nodes reachable from **s** in G'_f and return **S-{s}**

Proof of Claim

- **A**=**S**-{**s**} satisfies precedence constraints
  - No edge of **G** crosses forward out of **A** by our choice of capacities
  - Only forward edges cut are of the form (**v**,**t**) for **v**∈**A** or (**s**,**v**) for **v**∉**A**
  - The (**v**,**t**) edges for **v**∈**A** contribute

    $$\sum_{v \in A: p(v)<0} -p(v) = -\sum_{v \in A: p(v)<0} p(v)$$
  - The (**s**,**v**) edges for **v**∉**A** contribute

    $$\sum_{v \notin A: p(v) \geq 0} p(v) = C - \sum_{v \in A: p(v) \geq 0} p(v)$$
  - Therefore the total capacity of the cut is

    $$c(S,T) = C - \sum_{v \in A} p(v) = C - \text{Profit}(A)$$

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