CSE 421: Introduction to Algorithms

Network Flow

Winter 2005
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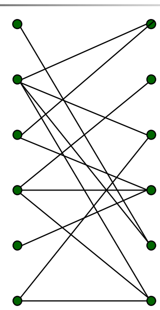
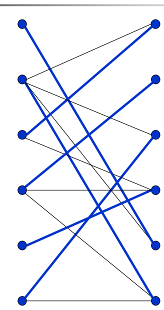
Bipartite Matching

- Given: A bipartite graph G=(V,E)
  - **M**⊆**E** is a matching in **G** iff no two edges in **M** share a vertex
- Goal: Find a matching **M** in **G** of maximum possible size

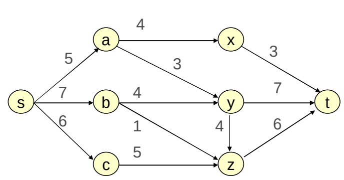
Bipartite Matching



Bipartite Matching

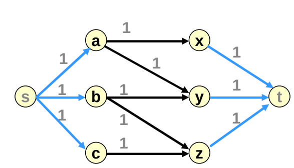


The Network Flow Problem



- How much stuff can flow from s to t?

Bipartite matching as a special case of flow



Net Flow: Formal Definition

Given:

A digraph **G** = (**V**,**E**)

Two vertices **s**,**t** in **V** (source & sink)

A *capacity* **c(u,v)** ≥ **0** for each (**u**,**v**) ∈ **E** (and **c**(**u**,**v**) = **0** for all non-edges (**u**,**v**))

Find:

A *flow function* **f**: **E** → **R** s.t., for all **u**,**v**:

- **0 ≤ f(u,v)** ≤ **c(u,v)** [Capacity Constraint]
- if **u** ≠ **s**,**t**, i.e. $f^{out}(u) = f^{in}(u)$ [Flow Conservation]

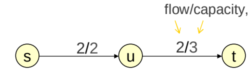
Maximizing total flow ν(**f**) = $f^{out}(s)$

Notation:

$$f^{in}(v) = \sum_{e=(u,v)\in E} f(u,v) \qquad f^{out}(v) = \sum_{e=(v,w)\in E} f(v,w)$$

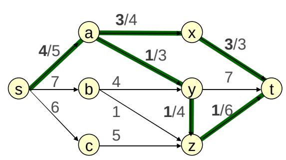
Example: A Flow Function

flow/capacity, not .66...



$f^{in}(u)=f(s,u)=2=f(u,t)=f^{out}(u)$

Example: A Flow Function



- Not shown: f(u,v) if = 0
- Note: max flow ≥ 4 since f is a flow function, with ν(**f**) = 4

Max Flow via a Greedy Alg?

While there is an s → t path in G

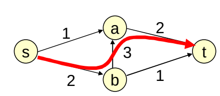
Pick such a path, **p**

Find **c**, the min capacity of any edge in **p**

Subtract **c** from all capacities on **p**

Delete edges of capacity **0**

- This does NOT always find a max flow:



If pick s →b →a →t first, flow stuck at 2. But flow 3 possible.

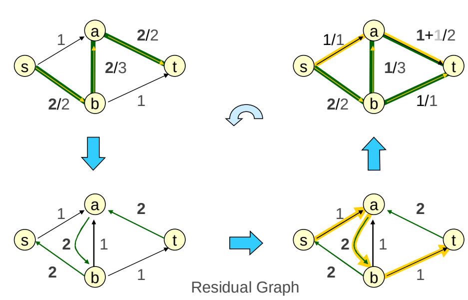
A Brief History of Flow

#	year	discoverer(s)	bound
1	1951	Dantzig	$O(n^2mU)$
2	1955	Ford & Fulkerson	$O(nmU)$
3	1970	Dinitz Edmonds & Karp	$O(nm^2)$
4	1970	Dinitz	$O(n^2m)$
5	1972	Edmonds & Karp Dinitz	$O(m^2 \log U)$
6	1973	Dinitz Gabow	$O(nm \log U)$
7	1974	Karzanov	$O(n^3)$
8	1977	Cherkassky	$O(n^2\sqrt{m})$
9	1980	Galil & Naamad	$O(nm \log^2 n)$
10	1983	Sleator & Tarjan	$O(nm \log n)$
11	1986	Goldberg & Tarjan	$O(nm \log(n^2/m))$
12	1987	Ahuja & Orlin	$O(nm + n^2 \log U)$
13	1987	Ahuja et al.	$O(nm \log(n\sqrt{\log U}/(m+2))$
14	1989	Cheriyan & Hagerup	$E(nm + n^2 \log^2 n)$
15	1990	Cheriyan et al.	$O(n^3/\log n)$
16	1990	Alon	$O(nm + n^{8/3} \log n)$
17	1992	King et al.	$O(nm + n^{2+\epsilon})$
18	1993	Phillips & Westbrook	$O(nm(\log_{m/n} n + \log^{2+\epsilon} n))$
19	1994	King et al.	$O(nm \log_{m/(n \log n)} n)$
20	1997	Goldberg & Rao	$O(m^{3/2} \log(n^2/m) \log U)$ $O(n^{2/3}m \log(n^2/m) \log U)$

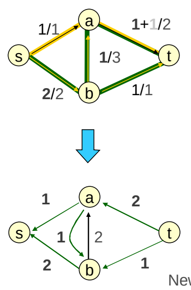
n = # of vertices
m= # of edges
U = Max capacity

Source: Goldberg & Rao, FOCS '97

Greed Revisited: Residual Graph & Augmenting Path



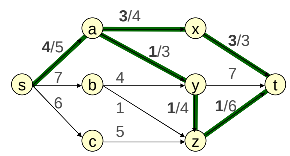
Greed Revisited: An Augmenting Path



New Residual Graph

Residual Capacity

- The *residual capacity* (w.r.t. **f**) of (**u**,**v**) is $c_f(u,v) = c(u,v) - f(u,v)$ if $f(u,v) \le c(u,v)$ and $c_f(u,v) = f(v,u)$ if $f(v,u) > 0$

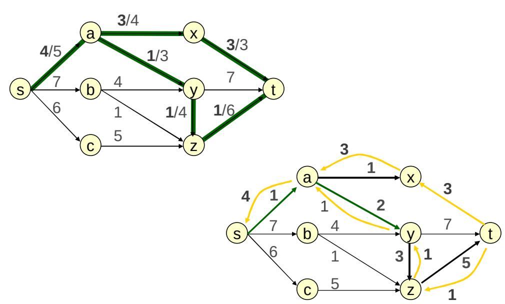


- e.g. $c_f(s,b)=7$; $c_f(a,x) = 1$; $c_f(x,a) = 3$

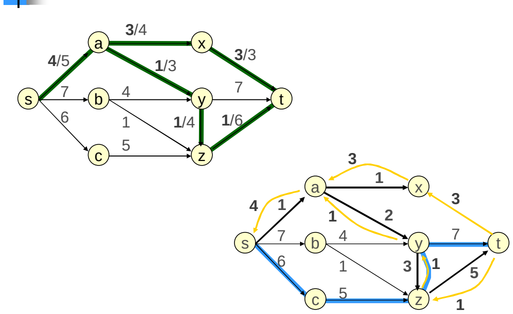
Residual Graph & Augmenting Paths

- The *residual graph* (w.r.t. **f**) is the graph $G_f = (V,E_f)$, where
  $E_f = \{ (u,v) \mid c_f(u,v) > 0 \}$
  - Two kinds of edges
    - Forward edges
      - $f(u,v) < c(u,v)$ so $c_f(u,v) = c(u,v) - f(u,v) > 0$
    - Backward edges
      - $f(u,v) > 0$ so $c_f(v,u) \ge -f(v,u) = f(u,v) > 0$
- An *augmenting path* (w.r.t. **f**) is a simple $s \to t$ path in G_f.

A Residual Network



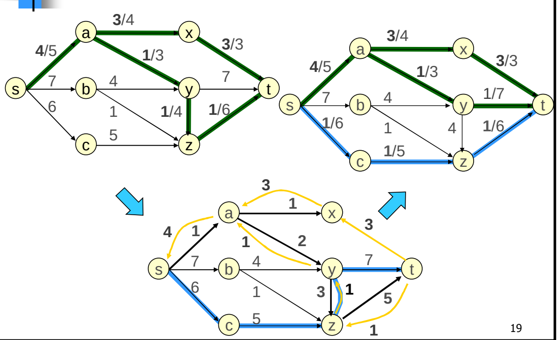
An Augmenting Path



Augmenting A Flow

augment(**f**,**P**)
  $c_P \leftarrow \min_{(u,v)\in P} c_f(u,v)$ "bottleneck(P)"
  for each $e \in P$
    if **e** is a forward edge then
      increase **f**(**e**) by c_P
    else (**e** is a backward edge)
      decrease **f**(**e**) by c_P
    endif
  endfor
  return(**f**)

Augmenting A Flow



Claim 7.1

If $\mathbf{G_f}$ has an augmenting path **P**, then the function **f'=**augment(**f**,**P**) is a legal flow.

Proof:

- **f'** and **f** differ only on the edges of **P** so only need to consider such edges (**u**,**v**)

Proof of Claim 7.1

- If (**u**,**v**) is a forward edge then
  $\mathbf{f'(u,v)=f(u,v)+c_P \le f(u,v)+c_f(u,v)}$
  $\mathbf{= f(u,v)+c(u,v)-f(u,v)}$
  $\mathbf{=c(u,v)}$
- If (**u**,**v**) is a backward edge then **f** and **f'** differ on flow along (**v**,**u**) instead of (**u**,**v**)
  $\mathbf{f'(v,u)=f(v,u)-c_P \ge f(v,u)-c_f(u,v)}$
  $\mathbf{= f(v,u)-f(v,u)=0}$
- Other conditions like flow conservation still met

Ford-Fulkerson Method

Start with **f**=**0** for every edge

While $\mathbf{G_f}$ has an augmenting path,
 augment

- Questions:
  - Does it halt?
  - Does it find a maximum flow?
  - How fast?

Observations about Ford-Fulkerson Algorithm

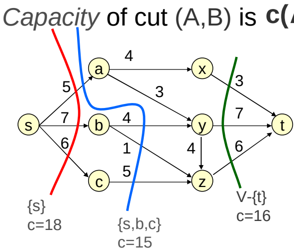
- At every stage the capacities and flow values are always integers (if they start that way)
- The flow value $\mathbf{v(f')=v(f)+c_P>v(f)}$ for **f'**=augment(**f**,**P**)
  - Since edges of residual capacity **0** do not appear in the residual graph
- Let $\mathbf{C=\Sigma_{(s,u)\in E}\ c(s,u)}$
  - $\mathbf{v(f) \le C}$
  - **F-F** does at most **C** rounds of augmentation since flows are integers and increase by at least **1** per step

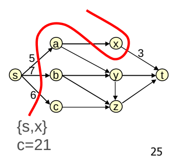
Running Time of Ford-Fulkerson

- For **f**=**0**, $\mathbf{G_f=G}$
- Finding an augmenting path in G_f is graph search O(**n**+**m**)=O(**m**) time
- Augmenting and updating $\mathbf{G_f}$ is O(**n**) time
- Total O(**mC**) time
- Does is find a maximum flow?
  - Need to show that for every flow **f** that isn't maximum $\mathbf{G_f}$ contains an **s**-**t**-path

Cuts

- A partition (A,B) of V is an *s-t-cut* if
  - $s \in A$, $t \in B$
- *Capacity* of cut (A,B) is $c(A,B) = \sum_{\substack{u \in A \\ v \in B}} c(u,v)$



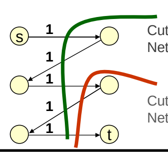


Convenient Definition

- $f^{out}(A)=\Sigma_{v\in A,\, w\notin A}\, f(v,w)$
- $f^{in}(A)=\Sigma_{v\in A,\, u\notin A}\, f(u,v)$

Claims 7.6 and 7.8

- For any flow f and any cut (A,B),
  - the net flow across the cut equals the total flow, i.e., $v(f) = f^{out}(A)-f^{in}(A)$, and
  - the net flow across the cut cannot exceed the capacity of the cut, i.e. $f^{out}(A)-f^{in}(A) \le c(A,B)$
- Corollary :
  Max flow ≤ Min cut



Proof of Claim 7.6

- Consider a set A with $s\in A$, $t\notin A$
- $f^{out}(A)-f^{in}(A) = \Sigma_{v\in A,\, w\notin A}\, f(v,w) - \Sigma_{v\in A,\, u\notin A}\, f(u,v)$
- We can add flow values for edges with both endpoints in A to **both** sums and they would cancel out so
- $$\begin{aligned} f^{out}(A)-f^{in}(A) &= \Sigma_{v\in A,\, w\in V}\, f(v,w) - \Sigma_{v\in A,\, u\in V}\, f(u,v) \\ &= \Sigma_{v\in A}\,(\Sigma_{w\in V}\, f(v,w) - \Sigma_{u\in V}\, f(u,v)) \\ &= \Sigma_{v\in A}\, f^{out}(v) - f^{in}(v) \\ &= f^{out}(s)-f^{in}(s) \end{aligned}$$

  since all other vertices have $f^{out}(v)=f^{in}(v)$
- $v(f) = f^{out}(s)$ and $f^{in}(s)=0$

Proof of Claim 7.8

- $$\begin{aligned} v(f) &= f^{out}(A)-f^{in}(A) \\ &\le f^{out}(A) \\ &= \Sigma_{v\in A,\, w\notin A}\, f(v,w) \\ &\le \Sigma_{v\in A,\, w\notin A}\, c(v,w) \\ &\le \Sigma_{v\in A,\, w\in B}\, c(v,w) \\ &= c(A,B) \end{aligned}$$

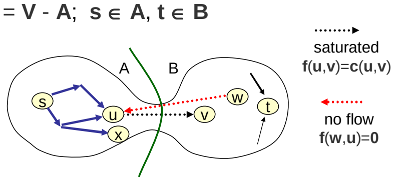
Max Flow / Min Cut Theorem

Claim 7.9 For any flow f, if G_f has no augmenting path then there is some s-t-cut (A,B) such that $v(f)=c(A,B)$ (proof on next slide)

- We know by Claims 7.6 & 7.8 that any flow f' satisfies $v(f') \le c(A,B)$ and we know that F-F runs for finite time until it finds a flow f satisfying conditions of Claim 7.9
  - Therefore by 7.9 for any flow f', $v(f') \le v(f)$
- Corollary (1) F-F computes a maximum flow in G
  (2) For any graph G, the value $v(f)$ of a maximum flow = minimum capacity $c(A,B)$ of any s-t-cut in G

Claim 7.9

Let **A** = { **u** | ∃ an path in $\mathbf{G_f}$ from **s** to **u** }
B = **V** - **A**; **s** ∈ **A**, **t** ∈ **B**

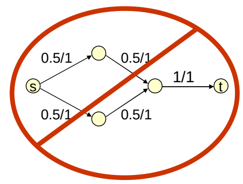


This is true for **every** edge crossing the cut, i.e.

$f^{out}(A) = \sum_{u\in A, v\in B} f(u,v) = \sum_{u\in A, v\in B} c(u,v) = c(A,B)$ and $f^{in}(A)=0$ so $v(f)=f^{out}(A)-f^{in}(A)=c(A,B)$

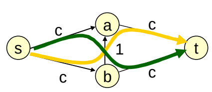
Flow Integrality Theorem

If all capacities are integers
- The max flow has an integer value
- Ford-Fulkerson method finds a max flow in which f(u,v) is an integer for all edges (u,v)



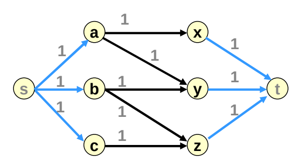
Corollaries & Facts

- If Ford-Fulkerson terminates, then it's found a max flow.
- It will terminate if **c**(**e**) integer or rational (but may not if they're irrational).
- However, may take exponential time, even with integer capacities:



c = 10^9, say

Bipartite matching as a special case of flow



Integer flows implies each flow is just a subset of the edges

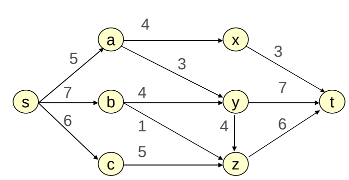
Therefore flow corresponds to a matching

O(**mC**)=O(**nm**) running time

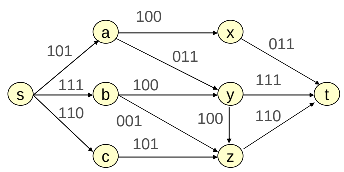
Capacity-scaling algorithm

- General idea:
  - Choose augmenting paths **P** with 'large' capacity $\mathbf{c_P}$
  - Can augment flows along a path **P** by any amount $\mathbf{b \leq c_P}$
    - Ford-Fulkerson still works
  - Get a flow that is maximum for the high-order bits first and then add more bits later

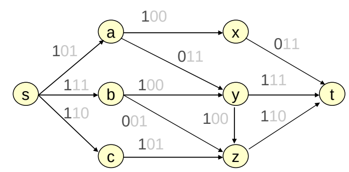
Capacity Scaling



Capacity Scaling

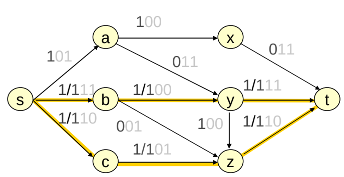


Capacity Scaling Bit 1



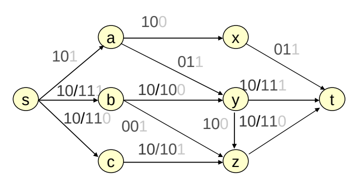
Capacity on each edge is at most **1**

Capacity Scaling Bit 1



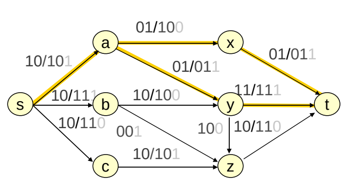
O(**nm**) time

Capacity Scaling Bit 2



Residual capacity across min cut is at most **m**

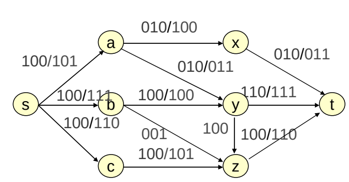
Capacity Scaling Bit 2



Residual capacity across min cut is at most **m**

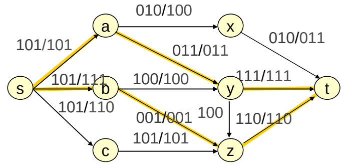
$\Rightarrow$ **O**(**m**) augmentations

Capacity Scaling Bit 3



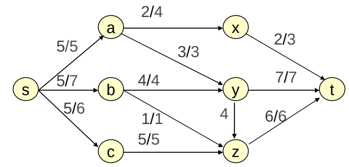
Residual capacity across min cut is at most **m**

Capacity Scaling Bit 3

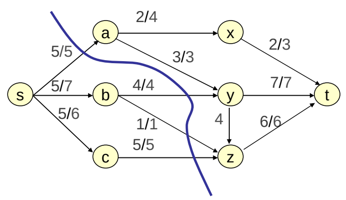


After O(**m**) augmentations

Capacity Scaling Final



Capacity Scaling Min Cut



Total time for capacity scaling

- $\log_2 \mathbf{U}$ rounds where **U** is largest capacity
- At most **m** augmentations per round
  - Let $\mathbf{c}_i$ be the capacities used in the $\mathbf{i}^{th}$ round and $\mathbf{f}_i$ be the maxflow found in the $\mathbf{i}^{th}$ round
    - For any edge $(\mathbf{u},\mathbf{v})$, $\mathbf{c}_{i+1}(\mathbf{u},\mathbf{v}) \le \mathbf{2c}_i(\mathbf{u},\mathbf{v})+\mathbf{1}$
  - $\mathbf{i+1}^{st}$ round starts with flow $\mathbf{f} = \mathbf{2}\ \mathbf{f}_i$
  - Let $(\mathbf{A},\mathbf{B})$ be a min cut from the $\mathbf{i}^{th}$ round
    - $\mathbf{v}(\mathbf{f}_i)=\mathbf{c}_i(\mathbf{A},\mathbf{B})$ so $\mathbf{v}(\mathbf{f})=\mathbf{2c}_i(\mathbf{A},\mathbf{B})$
  - $\mathbf{v}(\mathbf{f}_{i+1}) \le \mathbf{c}_{i+1}(\mathbf{A},\mathbf{B}) \le \mathbf{2c}_i(\mathbf{A},\mathbf{B})+\mathbf{m} = \mathbf{v}(\mathbf{f})+\mathbf{m}$
- O(**m**) time per augmentation
- Total time O($\mathbf{m}^2$ log **U**)

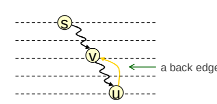
Edmonds-Karp Algorithm

- Use a shortest augmenting path
  (via Breadth First Search in residual graph)
- Time: O(**n m²**)

BFS/Shortest Path Lemmas

Distance from **s** in $\mathbf{G}_f$ is never reduced by:

- Deleting an edge
  Proof: no new (hence no shorter) path created
- Adding an edge (u,v), provided v is nearer than u
  Proof: BFS is unchanged, since **v** visited before **(u,v)** examined



Key Lemma

Let f be a flow, G_f the residual graph, and P a shortest augmenting path. Then no vertex is closer to **s** after augmentation along **P**.

Proof: Augmentation along **P** only deletes forward edges, or adds back edges that go to previous vertices along **P**

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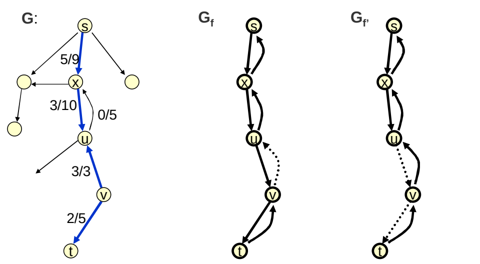
Augmentation vs BFS



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Theorem

The Edmonds-Karp Algorithm performs O(**mn**) flow augmentations

Proof:

Call (**u**,**v**) critical for augmenting path **P** if it's closest to **s** having min residual capacity

It will disappear from G_f after augmenting along **P**

In order for **(u,v)** to be critical again the **(u,v)** edge must re-appear in G_f but that will only happen when the distance to **u** has increased by **1**

It won't be critical again until farther from **s** so each edge critical at most **n** times

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Corollary

- Edmonds-Karp runs in O(**nm²**) time

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Project Selection a.k.a. The Strip Mining Problem

- Given
  - a directed acyclic graph **G**=(**V**,**E**) representing precedence constraints on tasks (a task points to its **predecessors**)
  - a profit value **p**(**v**) associated with each task **v**∈**V** (may be positive or negative)
- Find
  - a set **A**⊆**V** of tasks that is closed under predecessors, i.e. if (**u**,**v**)∈**E** and **u**∈**A** then **v**∈**A**, that maximizes Profit(**A**)=$\Sigma_{v\in A}$ **p**(**v**)

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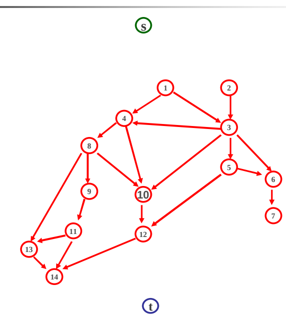
Extended Graph



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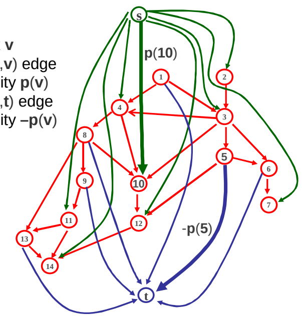
Extended Graph G'

For each vertex **v**
If **p**(**v**)≥**0** add (**s**,**v**) edge
with capacity **p**(**v**)
If **p**(**v**)<**0** add (**v**,**t**) edge
with capacity **–p**(**v**)



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Extended Graph G'

- Want to arrange capacities on edges of **G** so that for minimum **s**-**t**-cut (**S**,**T**) in **G'**, the set **A**=**S**-{**s**}
  - satisfies precedence constraints
  - has maximum possible profit in **G**
- Cut capacity with **S**={**s**} is just **C**=$\Sigma_{v:\ p(v)\geq 0}$ **p**(**v**)
  - Profit(**A**) ≤ **C** for any set **A**
- To satisfy precedence constraints don't want any original edges of **G** going forward across the minimum cut
  - That would correspond to a task in **A**=**S**-{**s**} that had a predecessor not in **A**=**S**-{**s**}
- Set capacity of each of these edges to **C+1**
  - The minimum cut has size at most **C**

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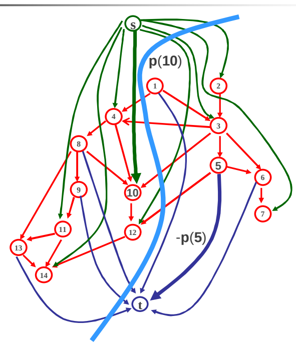
Extended Graph G'



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Project Selection

- **Claim** Any **s**-**t**-cut (**S**,**T**) in **G'** such that **A**=**S**-{**s**} satisfies precedence constraints has capacity

  c(**S**,**T**)=**C** - $\Sigma_{v\in A}$ **p**(**v**) = **C** - Profit(**A**)

- **Corollary** A minimum cut (**S**,**T**) in **G'** yields an optimal solution **A**=**S**-{**s**} to the profit selection problem
- **Algorithm** Compute maximum flow **f** in **G'**, find the set **S** of nodes reachable from **s** in G'_f and return **S**-{**s**}

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Proof of Claim

- **A**=**S**-{**s**} satisfies precedence constraints
  - No edge of **G** crosses forward out of **A** by our choice of capacities
  - Only forward edges cut are of the form (**v**,**t**) for **v**∈**A** or (**s**,**v**) for **v**∉**A**
  - The (**v**,**t**) edges for **v**∈**A** contribute

    $\Sigma_{v\in A: p(v)<0}$ -**p**(**v**) = - $\Sigma_{v\in A: p(v)<0}$ **p**(**v**)
  - The (**s**,**v**) edges for **v**∉**A** contribute

    $\Sigma_{v\notin A:\ p(v)\geq 0}$ **p**(**v**)=**C**-$\Sigma_{v\in A:\ p(v)\geq 0}$ **p**(**v**)
  - Therefore the total capacity of the cut is

    c(**S**,**T**) = **C** - $\Sigma_{v\in A}$ **p**(**v**) =**C**-Profit(**A**)

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