CSE 421
Intro to Algorithms
Summer 2004

Huffman Codes:
An Optimal Data Compression Method

Data Compression

- Binary character code ("code")
  - each k-bit source string maps to unique code word (e.g. k=8)
  - "compression" alg: concatenate code words for successive k-bit "characters" of source
- Fixed/variable length codes
  - all code words equal length?
- Prefix codes
  - no code word is prefix of another (simplifies decoding)

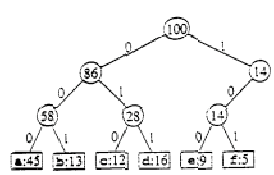
Compression Example

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

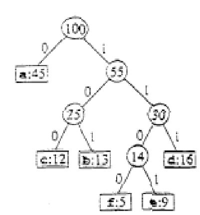
- 100k file, 6 letter alphabet:
- File Size:
  - ASCII, 8 bits/char: 800kbits
  - $2^3 > 6$; 3 bits/char: 300kbits
  - 00,01,10 for a,b,d; 11xx for c,e,f: 2.52 bits/char 74%*2 +26%*4: 252kbits
  - Optimal?
- Why?
  - Storage, transmission vs 1Ghz cpu

Prefix Codes = Trees

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%



$\underbrace{1\ 0\ 1}_{f}\ \underbrace{0\ 0\ 0}_{a}\ \underbrace{0\ 0\ 1}_{b}$

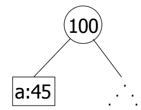


$\underbrace{1\ 1\ 0\ 0}_{f}\ \underbrace{0}_{a}\ \underbrace{1\ 0\ 1}_{b}$

Greedy Idea #1

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

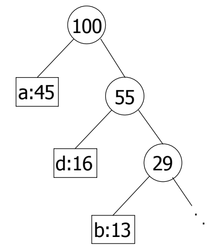
- Put most frequent under root, then recurse …



~~Greedy Idea #1~~

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

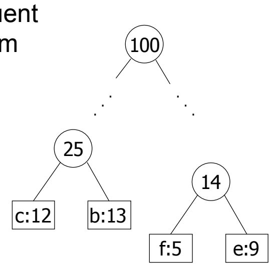
- ~~Put most frequent under root, then recurse …~~
- **Too greedy: unbalanced tree**

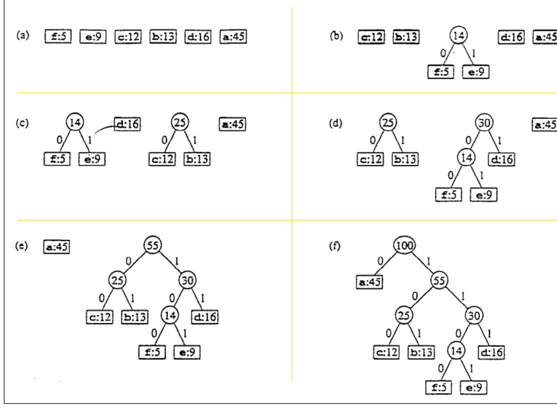


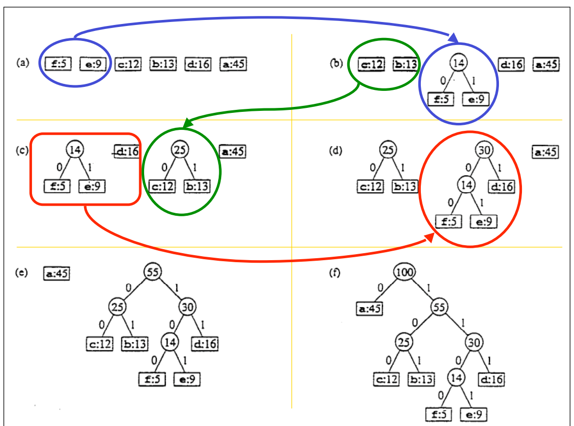
Greedy idea #2

a	45%
b	13%
c	12%
d	16%
e	9%
f	5%

- Group least frequent letters near bottom







Huffman's Algorithm (1952)

Algorithm:

insert node for each letter into priority queue by freq
while queue length > 1 do
    remove smallest 2; call them x, y
    make new node z from them, with f(z) = f(x)+f(y)
    insert z into queue

Analysis: O(n) heap ops: O(n log n)

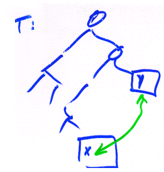
Goal: Minimize $B(T) = \sum_{c \in C} \text{freq}(c) * \text{depth}(c)$

Correctness: ???

Correctness Strategy

- Optimal solution may not be unique, so cannot prove that greedy gives the *only* possible answer.
- Instead, show that greedy's solution is as good as any.

Defn: A pair of leaves is an inversion if
depth(x) ≥ depth(y)
and
freq(x) ≥ freq(y)



Claim: If we flip an inversion, cost never increases.

Why? All other things being equal, better to give more frequent letter the shorter code.

$$\underbrace{(d(x)*f(x) + d(y)*f(y))}_{\text{before}} - \underbrace{(d(x)*f(y) + d(y)*f(x))}_{\text{after}} =$$

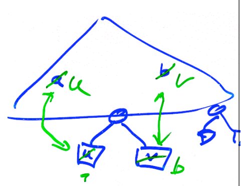
$$(d(x) - d(y)) * (f(x) - f(y)) \geq 0$$

I.e. non-negative cost savings.

Lemma 1: "Greedy Choice Property"

The 2 least frequent letters might as well be siblings at deepest level

- Let a be least freq, b 2nd
- Let u, v be siblings at max depth, f(u) ≤ f(v)
- Then (a,u) and (b,v) are inversions. Swap them.



Lemma 2: "Optimal Substructure"

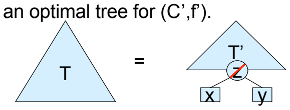
Let (C, f) be a problem instance: C an n-letter alphabet with letter frequencies f(c) for c in C.

For any x, y in C, let C' be the (n-1) letter alphabet C - {x,y} ∪ {z} and for all c in C' define

$$f'(c) = \begin{cases} f(c), & \text{if } c \neq x,y,z \\ f(x)+f(y), & \text{if } c = z \end{cases}$$

Let T' be an optimal tree for (C',f').

Then



is optimal for (C,f) among all trees having x,y as siblings

Proof:

$$B(T) = \sum_{c \in C} d_T(c) \cdot f(c)$$

$$\begin{aligned} B(T) - B(T') &= d_T(x) \cdot (f(x) + f(y)) - d_{T'}(z) \cdot f'(z) \\ &= (d_{T'}(z) + 1) \cdot f'(z) - d_{T'}(z) \cdot f'(z) \\ &= f'(z) \end{aligned}$$

Suppose $\hat{T}$ (having x & y as siblings) is better than T, i.e.

$B(\hat{T}) < B(T)$. Collapse x & y to z, forming $\hat{T}'$; as above:

$$B(\hat{T}) - B(\hat{T}') = f'(z)$$

Then:

$$B(\hat{T}') = B(\hat{T}) - f'(z) < B(T) - f'(z) = B(T')$$

Contradicting optimality of T'

Theorem: Huffman gives optimal codes

Proof: induction on |C|

- Basis: n=1,2 – immediate
- Induction: n>2
  - Let x,y be least frequent
  - Form C', f', & z, as above
  - By induction, T' is opt for (C',f')
  - By lemma 2, T'→T is opt for (C,f) among trees with x,y as siblings
  - By lemma 1, some opt tree has x, y as siblings
  - Therefore, T is optimal.

Data Compression

- Huffman is optimal.
- BUT still might do better!
  - Huffman encodes fixed length blocks. What if we vary them?
  - Huffman uses one encoding throughout a file. What if characteristics change?
  - What if data has structure? E.g. raster images, video,…
  - Huffman is lossless. Necessary?
- LZW, MPEG, …