CSE 421:
Introduction to Algorithms

Dynamic Programming

"Dynamic Programming"

Program — A plan or procedure for dealing with some matter – Webster's New World Dictionary

Dynamic Programming

- Outline:
  - Example 1 – Licking Stamps
  - General Principles
  - Example 2 – Knapsack (§ 5.10)
  - Example 3 – Sequence Comparison (§ 6.8)

Licking Stamps

- Given:
  - Large supply of 5¢, 4¢, and 1¢ stamps
  - An amount N
- Problem: choose fewest stamps totaling N

How to Lick 27¢

# of 5¢ Stamps	# of 4¢ Stamps	# of 1¢ Stamps	Total Number
5	0	2	7
4	1	3	8
3	3	0	6

Moral: Greed doesn't pay

A Simple Algorithm

- At most N stamps needed, etc.

```
for a = 0, …, N {
  for b = 0, …, N {
    for c = 0, …, N {
      if (5a+4b+c == N && a+b+c is new min)
        {retain (a,b,c);}}}
output retained triple;
```

- Time: $O(N^3)$
  (Not too hard to see some optimizations, but we're after bigger fish…)

Better Idea

Theorem: If last stamp licked in an optimal solution has value v, then previous stamps form an optimal solution for N-v.

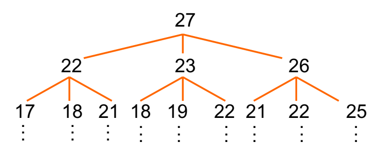
Proof: if not, we could improve the solution for N by using opt for N-v.

$$M(i) = \min\left\{\begin{array}{ll}0 & i=0\\ 1+M(i-5) & i\ge 5\\ 1+M(i-4) & i\ge 4\\ 1+M(i-1) & i\ge 1\end{array}\right\}$$

where *M(i)* = min number of stamps totaling *i*¢

New Idea: Recursion

$$M(i) = \min\left\{\begin{array}{ll}0 & i=0\\ 1+M(i-5) & i\ge 5\\ 1+M(i-4) & i\ge 4\\ 1+M(i-1) & i\ge 1\end{array}\right\}$$



Time: $> 3^{N/5}$

Another New Idea: Avoid Recomputation

- Tabulate values of solved subproblems
  - Top-down: "memoization"
  - Bottom up:

for i = 0, …, N do $M[i] = \min\left\{\begin{array}{ll}0 & i=0\\ 1+M[i-5] & i\ge 5\\ 1+M[i-4] & i\ge 4\\ 1+M[i-1] & i\ge 1\end{array}\right\}$;

- Time: O(N)

Finding How Many Stamps

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

1+Min(3,1,3) = 2

Finding Which Stamps: Trace-Back

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

4¢

1+Min(3,1,3) = 2

Complexity Note

- O(N) is better than $O(N^3)$ or $O(3^{N/5})$
- But still *exponential* in input size (log N bits)

  (E.g., miserably slow if N is 64 bits – $c \cdot 2^{64}$ steps for 64 bit input.)
- Note: can do in O(1) for 5¢, 4¢, and 1¢ but not in general. See "NP-Completeness" later

Elements of Dynamic Programming

- What feature did we use?
- What should we look for to use again?

- "Optimal Substructure"
  - Optimal solution contains optimal subproblems
- "Repeated Subproblems"
  - The same subproblems arise in various ways

The Knapsack Problem (§ 5.10)

Given positive integers W, $w_1, w_2, \ldots, w_n$,
Find a subset of the w_i's totaling exactly W.
Alternate (Easier?) Problem: Is there one?

(Like stamp problem, but limited supply of each.)

Motivation: simple 1-d abstraction of packing boxes, trucks, VLSI chips, …

Knapsack Example

$w_1, \ldots, w_4$ = 2, 5, 9, 11

- W = 14
  - YES: 5+9 = 14
- W = 15
  - NO:
    - all singletons 11: too small
    - all pairs too small, except 9+11, 5+11 too big
    - all triples 16: too big
    - all quadruples: too big

2^n possibilities

Solve by Induction? Try 1

- Defn: Let P(i) be true iff there is a subset of first i weights $w_1, w_2, \ldots, w_i$ totaling W
- Assume we know how to evaluate P(n-1)
  - Case 1: P(n-1) = True – done; w_n unneeded
  - Case 2: P(n-1) = False – may or may not be a solution, but if there is one, it *in*cludes w_n, and other included weights total W-w_n, but I.H. doesn't tell us how to find it. ☹

Solve by Induction? Try 2

- Defn: Let P(i, X) be true iff there is a subset of first i weights $w_1, w_2, \ldots, w_i$ totaling X
- Assume we know P(n-1, X) for all X W
  - Case 1: P(n-1, W) = True – done; w_n unneeded
  - Case 2: P(n-1, W) = False – may or may not be a solution, but if there is one, it *in*cludes w_n, and other weights total W-w_n, so P(n, W) = P(n-1, W-w_n) ☺
- Algorithm:
  - P(n,W) = P(n-1, W) ∨ (P(n-1, W-w_n) if W-w_n 7 0)
  - Basis: P(0, X) = True iff (X == 0)

Knapsack Example

P(n,W) = P(n-1, W) ∨ P(n-1, W-w_n)

$w_1, \ldots, w_4$ = 2, 5, 9, 11 W=15

i\X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0
3	1	0	1	0	0	1	0	1	0	1	0	1	0	0	1	0
4	1	0	1	0	0	1	0	1	0	1	0	1	0	1	1	0

W = 14: Yes
W = 15: No

Dynamic Programming?

$$P(n,W) = P(n-1, W) \vee P(n-1, W-w_n)$$

- Optimal substructure?

  Best/only way to fill a big knapsack implicitly fills smaller ones with fewer objects in the best or only way

- Repeated subproblems?

  Smallest cases potentially common to many bigger instances

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Complexity Notes

- Time is O(N W)
- May or may not beat naïve 2^N
- But still partially *exponential* in input size (N log W bits)
  - E.g., 100 weights, 64 bits each – 100 • 2^{64} array elements.
  - C.v., e.g., Skyline 100 bldgs, 64 bit coords – c • 100 • log 100 steps.
- See “NP-Completeness” later

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