CSE 421: Intro to Algorithms

Summer 2004

Graph Algorithms:
BFS, DFS, Articulation Points

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Breadth-First Search

- Completely explore the vertices in order of their distance from v
- Naturally implemented using a queue
- Works on general graphs, not just trees

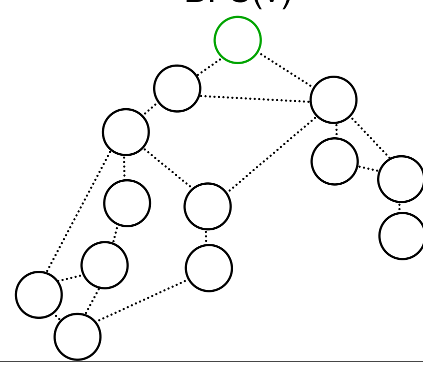
BFS(v)

Global initialization: mark all vertices "undiscovered"

```
BFS(v)
    mark  v "discovered"
    queue = v
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u completed
```

Exercise: modify code to number vertices & compute level numbers

BFS(v)



BFS analysis

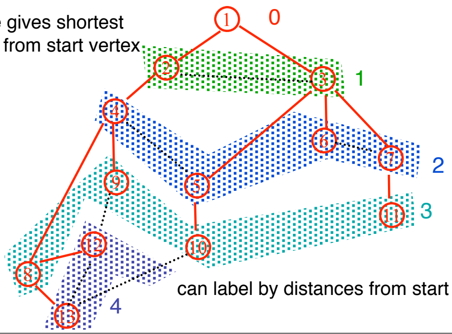
- Each edge is explored once from each end-point
- Each vertex is discovered by following a different edge
- Total cost O(m) where m=# of edges
- Disconnected? Restart @ undiscovered vertices: O(m+n)

Properties of (Undirected) BFS(v)

- BFS(v) visits x if and only if there is a path in G from v to x.
- Edges into then-undiscovered vertices define a ***tree*** – the "breadth first spanning tree" of G
- Level i in this tree are exactly those vertices u such that the shortest path (in G, not just the tree) from the root v is of length i.
- ***All*** non-tree edges join vertices on the same or adjacent levels

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



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Depth-First Search

- Follow the first path you find as far as you can go
- Back up to last unexplored edge when you reach a dead end, then go as far you can

- Naturally implemented using recursive calls or a stack
- Works on general graphs, not just trees

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DFS(v) – Recursive version

Global Initialization:
  mark all vertices v "undiscovered" via v.dfs# = -1
  dfscounter = 0

```
DFS(v)
    v.dfs# = dfscounter++    // mark  v "discovered"
    for each edge (v,x)
      if (x.dfs# = -1)       // tree edge (x previously  undiscovered)
           DFS(x)
      else …                 // code for back-, fwd-, parent,
                             // edges, if needed
    // mark v "completed," if needed
```

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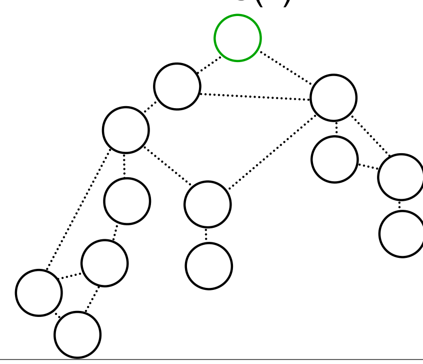
DFS(v)



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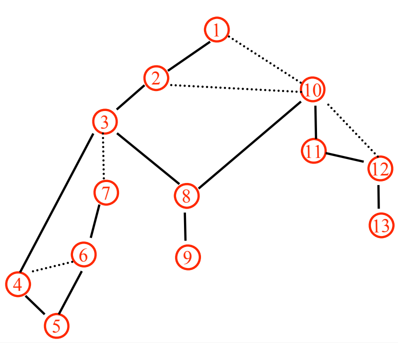
DFS(v)



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Properties of (Undirected) DFS(v)

- Like BFS(v):
  - DFS(v) visits x ⇔ there is a path in G from v to x (through previously unvisited vertices)
  - Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G
- Unlike the BFS tree:
  - the DF spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

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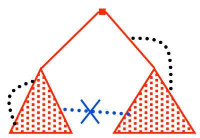
Non-tree edges

- All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree
- Called back/forward edges (depending on end)
- No cross edges!



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Application: Articulation Points

- A node in an undirected graph is an **articulation point** iff removing it disconnects the graph
- articulation points represent vulnerabilities in a network – single points whose failure would split the network into 2 or more disconnected components

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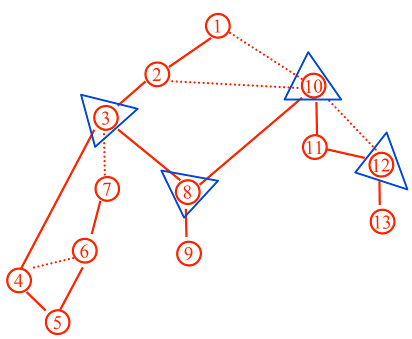
Articulation Points



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Exercise

- draw a graph, ~ 10 nodes, A-J
- redraw as via DFS
- add dsf#s & tree/back edges (solid/dashed)
- find cycles
- give alg to find cycles via dfs; does G have any?
- find articulation points
- what do cycles have to do with articulation points?
- alg to find articulation points via DFS???

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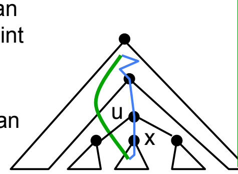
Articulation Points from DFS

- Root node is an articulation point iff it has more than one child
- Leaf is never an articulation point
- non-leaf, non-root node u is an articulation point ⇔ no non-tree edge goes above u from a sub-tree below some child of u



If removal of u does NOT separate x, there must be an exit from x's subtree. How? Via back edge.

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Articulation Points: the "LOW" function

- Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or connected to a vertex in that subtree by a back edge.
- Key idea 1: if some child x of v has LOW(x) ≥ dfs#(v) then v is an articulation point.
- Key idea 2: LOW(v) = min ({LOW(w) | w a child of v } ∪ { dfs#(x) | {v,x} is a back edge from v })

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Properties of DFS Vertex Numbering

- If u is an ancestor of v in the DFS tree,

  then

  ?

  dfs#(u) $\boxed{}$ dfs#(v).

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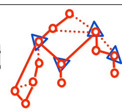
DFS(v) for Finding Articulation Points

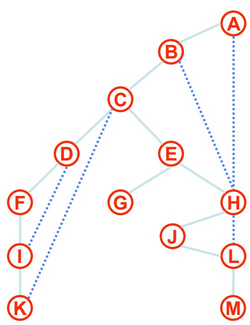
```
Global initialization: v.dfs# = -1 ∀v; DFS(v) ∀unvisited v.
DFS(v)
 v.dfs# = dfscounter++
 v.low = v.dfs#             // initialization
 for each edge {v,x}
      if (x.dfs# == -1)     // x is undiscovered
         DFS(x)
         v.low = min(v.low, x.low)
         if (x.low >= v.dfs#)
            print "v is art. pt., separating x"
      else if (x is not v's parent)
         v.low = min(v.low, x.dfs#)
```

Except for root. Why?

Equiv: "if({v,x} is a back edge)" Why?

Articulation Points





Vertex	DFS #	Low
A		
B		
C		
D		
E		
F		
G		
H		
I		
J		
K		
L		
M		

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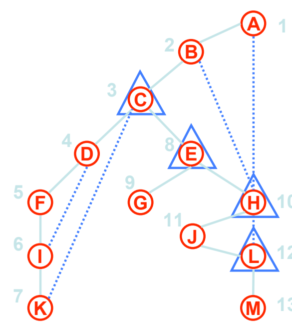
Articulation Points



Vertex	DFS #	Low
A	1	1
B	2	1
C	3	1
D	4	3
E	8	1
F	5	3
G	9	9
H	10	1
I	6	3
J	11	10
K	7	3
L	12	10
M	13	13

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