CSci 421
Introduction to Algorithms
Homework Assignment 6
Due: Tuesday, 10 Aug 2004

Reading: Chapter 11

1. (a) Draw the residual graph corresponding to the flow in figure 7.41, pg241. Is this flow maximum? Why or why not? If maximum, what is the corresponding min cut?

   (b) Repeat part (a) assuming $c(v, u)$ = 6 (instead of 4, as shown in the figure).

2. Let $G = (V, E)$ be a directed graph with edge capacities given by $c : E \to \Re^+$ (the non-negative reals), $f : V \times V \to \Re$ be a flow on G (as defined in lecture; I think you'll find it simpler to work with than the definition on page 238). Let G_f be the residual graph induced by f. Finally let $g : V \times V \to \Re$ be a flow function on G_f (not G), and define $h : V \times V \to \Re$ to be $f + g$, i.e. for all $u, v \in V$, $h(u, v) = f(u, v) + g(u, v)$.

   Prove or disprove: h is a flow on G.

   Note: I showed in lecture that this result is true in the special case where g sends a non-zero flow only along a single s–t path, so the question here is whether that generalizes to an arbitrary augmenting flow.

3. Note: In this prob. and the next, as in lecture, I use the terms *alternating* and *augmenting* path slightly differently from the book. A path is *alternating* with respect to a given matching M if its edges alternate between M and $E - M$. An *augmenting* path is an alternating path whose end points are both unmatched. Compare to the book's definition on page 236.

   Let G be the bipartite graph shown in figure 7.37, page 236. Let M be the (non-maximum) matching $\{ \{3, A\}, \{4, E\}, \{6, F\}\}$.

   (a) List 3 alternating paths that are *not* augmenting paths.

   (b) List *all* augmenting paths in G (with respect to M).

   (c) What is the smallest maximal set of pairwise vertex-disjoint augmenting paths? What is the largest?

   Terminology: two paths are "vertex disjoint" if they don't have any vertices in common. A set of paths is "pairwise vertex disjoint" if no two of the paths in the set have any vertices in common. The full set of augmenting paths is not pairwise vertex disjoint, but various subsets of it are. Such a subset is "maximal" if it can't be enlarged without destroying the "pairwise vertex disjoint" property. So the question is asking you to give the largest (most paths) and smallest such sets.

   (d) Let P be the augmenting path of length 3 containing $\{4, E\}$. Considering M and P to be sets of edges, $M \oplus P$ is their set theoretic *symmetric difference*: $(M \cup P) - (M \cap P)$. What set of edges is $M' = M \oplus P$? Is it a matching?

4. Let G be any bipartite graph, M any matching in G, and P any augmenting path (with respect to M).

   (a) Prove that $M' = M \oplus P$ is a matching.

   (b) Show $|M'| = |M| + 1$. How is the set of matched vertices in M' related to the set of matched vertices in M and the set of vertices (incident to edges) in P?

   (c) Give a counterexample to 4a if P is an arbitrary path, i.e. show that there is a graph G, matching M and path P such that $M \oplus P$ is not a matching. Is it true or false if P is an alternating path that is not an augmenting path? Prove or give a counterexample.

   (d) Suppose that there are *two* augmenting paths P and P' with respect to M, and that P and P' are vertex-disjoint. Show that P' also is an augmenting path with respect to the *augmented* matching $(M \oplus P)$, and similarly that P is augmenting with respect to $(M \oplus P')$. What could you say about a case where there were, say, 17 pairwise disjoint paths $P_1, \ldots, P_{17}$, all augmenting paths with respect to M? What, and how big, is $M \oplus P_1 \oplus \cdots \oplus P_{17}$?

5. The Hopcroft-Karp bipartite matching algorithm sketched in class and the book needs a subroutine for the following problem: Given a directed acyclic graph G with a designated set U of vertices having indegree 0 (the *source* vertices) and a designated set V of vertices having outdegree 0 (the *sinks*), find a maximal set of pairwise vertex disjoint paths that go from some source to some sink. Give a linear time algorithm for this problem.

   [Note that in the matching example the graph G has the additional property that, since it is produced by breadth-first search, it is nicely *layered* — each vertex has been assigned a layer number with all sources on layer 0, all sinks on layer k for some fixed k, and all edges going from a layer i to the next layer $i + 1$. I don't think this extra information is either necessary or particularly useful in solving the problem, BUT you may assume it if you find it helpful.]

6. (Optional Extra Credit:) Text, 7.98.

7. (Optional Extra Credit:) A graph is d-*regular* if every vertex has degree exactly d. Prove that a d-regular bipartite graph always has a perfect matching. Hints: How many vertices are in the left and right sides? Remember the max flow/min cut theorem.