CSE 421
Introduction to Algorithms
Winter 2000

NP-Completeness

(Chapter 11)

Easy Problems vs. Hard Problems

Easy - problems whose worst case running time is bounded by some **polynomial** in the size of the input.

Easy = Efficient

Hard - problems that *cannot* be *solved efficiently*.

The class P

Definition: P = set of problems solvable by computers in polynomial time.

i.e. $T(n) = O(n^k)$ for some k.

- These problems are sometimes called **tractable** problems.

Examples: sorting, SCC, matching, max flow, shortest path, MST *– all of 421 except Stamps/Knapsack/Partition*

Is P a good definition of efficient?

Is $O(n^{100})$ efficient? Is $O(10^9 n)$ efficient?

Are $O(2^n)$, $O(2^{n/1000})$, $O(n^{\log n})$, …really so bad?

So we have:

P = "easy" = efficient = tractable
= solvable in polynomial-time

not P = hard = not tractable

USUALLY

Decision Problems

- Technically, we restrict discussion to **decision problems** - problems that have an answer of either yes or no.
- Usually easy to convert to decision problem:
  - **Example**: Instead of looking for the size of the shortest path from *s* to *t* in a graph *G*, we ask:

    "Is there a path from *s* to *t* of length $\leq k$?"

Examples of Decision Problems in P

Big Flow

Given: graph *G* with edge lengths, vertices *s* and *t,* integer *k.*

Question: Is there an *s-t* flow of length $\geq k$?

Small Spanning Tree

Given: weighted undirected graph *G*, integer *k*.

Question: Is there a spanning tree of weight $\leq k$?

Decision Problems

Loss of generality?

A. Not much. If we know how to solve the decision problem, then we can usually solve the original problem.

B. More importantly, decision problem is easier (at least, not harder), so a lower bound on decision problem is a lower bound on general problem.

Decision problem as a Language-recognition problem

- Let *U* be the set of all possible inputs to the decision problem.
- $L \subseteq U$ = the set of all inputs for which the answer to the problem is **yes**.
- We call *L* the **language** corresponding to the problem. (problem = language)
- The decision problem is thus:
  - to recognize whether or not a given input belongs to *L* = the language recognition problem.

The class NP

Definition: NP = set of problems solvable by a *nondeterministic* algorithm in polynomial time.

Another way of saying this:

NP = The class of problems whose solution can be **verified** in polynomial time.

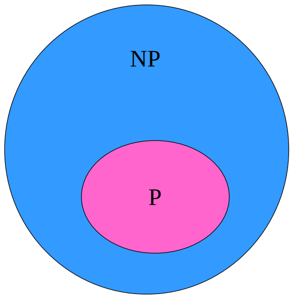
NP = "nondeterministic polynomial"

Examples: all of problems in P plus: SAT, TSP, Hamiltonian cycle, bin packing, vertex cover.

Complexity Classes

NP = Polynomial-time **verifiable**

P = Polynomial-time **solvable**



Verifying Solutions

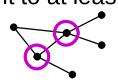
Given a **problem** and a **potential solution**, verify that the solution is correct in polynomial-time.

In general, **guess** a solution, and then **check** if the guess is correct in polynomial time.

Examples of Problems in NP

Vertex Cover

A **vertex cover** of G is a set of vertices such that every edge in G is incident to at least one of these vertices. Example:



Question: Given a graph G, integer k, determine whether G has a vertex cover containing $\leq k$ vertices?

Verify: Given a set of $\leq k$ vertices, does it cover every edge? (Guess and check in polynomial time.)

Examples of Problems in NP

Satisfiability (SAT)

A Boolean formula in conjunctive normal form (CNF) is **satisfiable** if there exists a truth assignment of 0's and 1's to its variables such that the value of the expression is 1. Example:

$$S=(x+y+\neg z)\bullet(\neg x+y+z)\bullet(\neg x+\neg y+\neg z)$$

Question: Given a Boolean formula in CNF, is it satisfiable?

Verify: Given a truth assignment, does it satisfy the formula? (Guess and check in polynomial time.)

Problems in P can also be verified in polynomial-time

Shortest Path: Given a graph G with edge lengths, is there a path from s to t of length $\leq k$?

Verify: Given a path from s to t, is its length $\leq k$?

Small Spanning Tree: Given a weighted undirected graph G, is there a spanning tree of weight $\leq k$?

Verify: Given a spanning tree, is its weight $\leq k$?

Nondeterminism

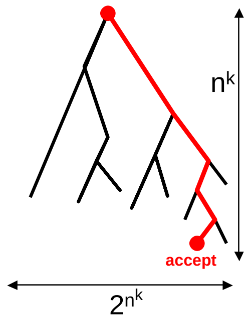
- A **nondeterministic algorithm** has all the "regular" operations of any other algorithm available to it.
- *In addition*, it has a powerful primitive, the **nd-choice primitive**.
- The **nd-choice primitive** is associated with a fixed number of choices, such that each choice causes the algorithm to follow a different computation path.

Nondeterminism (cont.)

- A **nondeterministic algorithm** consists of an interleaving of regular deterministic steps and uses of the **nd-choice primitive**.
- Definition: the algorithm accepts a language L if and only if
  - It has at least one "good" (accepting) sequence of choices for every $x \in L$, and
  - For all $x \notin L$, it reaches a reject outcome on **all** paths.

P vs NP vs Exponential Time

- Theorem: Every problem in NP can be solved deterministically in exponential time
- Proof: the nondeterministic algorithm makes only n^k nd-choices. Try all 2^{n^k} possibilities; if any succeed, accept; if all fail, reject.



The class NP-complete

We are pretty sure that no problem in NP – P can be solved in polynomial time.

Non-Definition: NP-complete = the **hardest** problems in the class NP. (Formal definition later.)

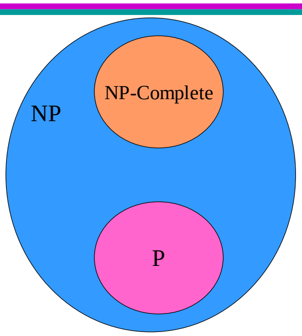
Interesting fact: If any one NP-complete problem could be solved in polynomial time, then ***all*** NP-complete problems could be solved in polynomial time.

Complexity Classes

NP = Poly-time **verifiable**

P = Poly-time **solvable**

NP-Complete = "Hardest" problems in **NP**



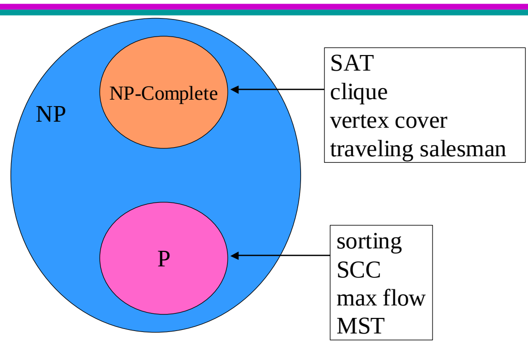
The class NP-complete (cont.)

Thousands of important problems have been shown to be NP-complete.

Fact (Dogma): The general belief is that there is no efficient algorithm for any **NP-complete** problem, but no proof of that belief is known.

Examples: SAT, clique, vertex cover, Hamiltonian cycle, TSP, bin packing.

Complexity Classes of Problems



Does P = NP?

- This is an open question.
- To show that **P = NP**, we have to show that *every* problem that belongs to NP can be solved by a polynomial time deterministic algorithm.
- No one has shown this yet.
- (It seems unlikely to be true.)

Is all of this useful for anything??!?

Earlier in this class we learned techniques for solving problems in **P**.

Question: Do we just throw up our hands if we come across a problem we suspect **not to be in P**?

Dealing with NP-complete Problems

What if I think my problem is <u>not in P</u>?

<u>Here is what you might do</u>:

1) Prove your problem is **NP-complete**
(a common, but not guaranteed outcome)

2) Come up with an algorithm to solve the problem **usually** or **approximately**.

Reductions: a useful tool

Definition: To **reduce** A to B means to figure out how to solve A, given a subroutine solving B.

Example: reduce MEDIAN to SORT

Solution: sort, then select (n/2) th

Example: reduce SORT to FIND_MAX

Solution: FIND_MAX, remove it, repeat

Example: reduce MEDIAN to FIND_MAX

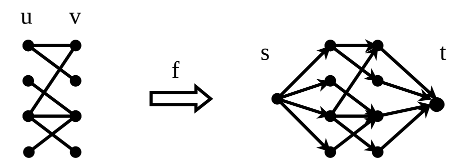
Solution: transitivity: compose solutions above.

More Examples of reductions

Example:

reduce BIPARTITE_MATCHING to MAX_FLOW

Is there a matching of size k? Is there a flow of size k?



All capacities = 1

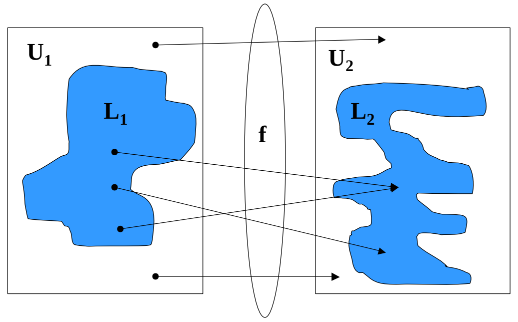
Polynomial-Time Reductions

Definition: Let L_1 and L_2 be two languages from the input spaces U_1 and U_2.

We say that L_1 is **polynomially reducible** to L_2 if there exists a polynomial-time algorithm f that converts each input $u_1 \in U_1$ to another input $u_2 \in U_2$ such that $u_1 \in L_1$ iff $u_2 \in L_2$.

$$u_1 \in L_1 \Leftrightarrow f(u_1) \in L_2$$

Polynomial-time Reduction from language L_1 to language L_2 via reduction function f.



$$u_1 \in L_1 \Leftrightarrow f(u_1) \in L_2$$

Polynomial-Time Reductions (cont.)

Define: $\boldsymbol{A \le_p B}$ *"A is polynomial-time reducible to B", iff there is a polynomial-time computable function f such that:* $\boldsymbol{x \in A \Leftrightarrow f(x) \in B}$

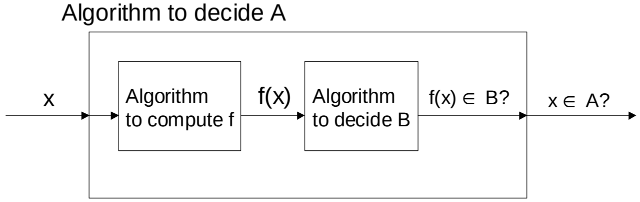
"complexity of A" ≤ *"complexity of B" + "complexity of f"*

(1) $\boldsymbol{A \le_p B}$ *and* $\boldsymbol{B \in P \Rightarrow A \in P}$

(2) $\boldsymbol{A \le_p B}$ *and* $\boldsymbol{A \notin P \Rightarrow B \notin P}$

(3) $\boldsymbol{A \le_p B}$ *and* $\boldsymbol{B \le_p C \Rightarrow A \le_p C}$ *(transitivity)*

Using an Algorithm for *B* to Decide *A*



"If A $\le_p$ B, and we can solve B in polynomial time, then we can solve A in polynomial time also."

Ex: suppose f takes $O(n^3)$ and algorithm for B takes $O(n^2)$. How long does the above algorithm for A take?

Definition of NP-Completeness

Definition: Problem *B* is **NP-hard** if *every* problem in NP is polynomially reducible to *B*.

Definition: Problem *B* is **NP-complete** if:

(1) *B* belongs to NP, and

(2) *B* is NP-hard.

Proving a problem is NP-complete

- Technically,for condition (2) we have to show that **every** problem in NP is reducible to B. (yikes!) This sounds like a lot of work.
- For the **very first NP-complete problem** (SAT) this had to be proved directly.
- However, once we have one NP-complete problem, then we don't have to do this every time.
- Why? Transitivity.

Re-stated Definition

Lemma 11.3: Problem *B* is **NP-complete** if:

(1) *B* belongs to NP, and

(2') *A* is polynomial-time reducible to *B*, for some problem *A* that is NP-complete.

That is, to show (2') given a new problem *B*, it is sufficient to show that SAT or any other NP-complete problem is polynomial-time reducible to *B*.

Usefulness of Transitivity

Now we only have to show $L' \leq_p L$, for some problem $L' \in$ ***NP-complete***, in order to show that **L** is NP-hard. Why is this equivalent?

1) Since $L' \in$ ***NP-complete****, we know that* ***L'*** is *NP-hard.* That is:

$$\forall L'' \in NP, \text{ we have } L'' \leq_p L'$$

2) If we show $L' \leq_p L$, then by transitivity we know that: $\forall L'' \in NP$, we have $L'' \leq_p L$.

Thus L is NP-hard.

The growth of the number of NP-complete problems

- Steve Cook (1971) showed that SAT was NP-complete.
- Richard Karp (1972) found 24 more NP-complete problems.
- Today there are thousands of known NP-complete problems.
  - Garey and Johnson (1979) is an excellent source of NP-complete problems.

SAT is NP-complete

Cook's theorem: SAT is NP-complete

Satisfiability (SAT)

A Boolean formula in conjunctive normal form (CNF) is **satisfiable** if there exists a truth assignment of 0's and 1's to its variables such that the value of the expression is 1. Example:

$$S=(x+y+\neg z)\bullet(\neg x+y+z)\bullet(\neg x+\neg y+\neg z)$$

Example above is satisfiable. (We an see this by setting x=1, y=1 and z=0.)

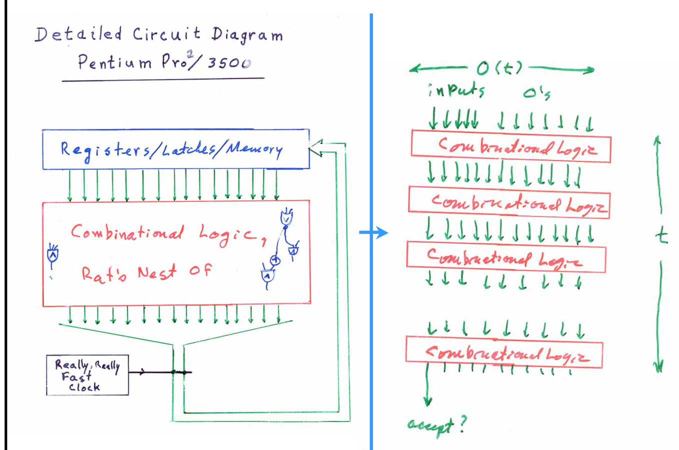
SAT is NP-complete

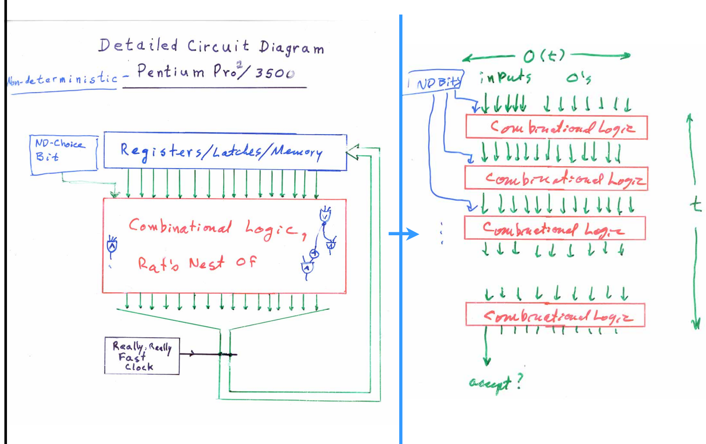
Rough idea of proof:

(1) **SAT is in NP** because we can guess a truth assignment and check that it satisfies the expression in polynomial time.

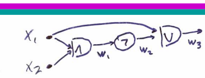
(2) **SAT is NP-hard** because …..

Cook proved it directly, but easier to see via an intermediate problem – **Circuit-SAT**





Circuit-SAT $\leq_p$ 3-SAT



$(w_1 \leftrightarrow (x_1 \wedge x_2)) \wedge (w_2 \leftrightarrow (\neg w_1)) \wedge (w_3 \leftrightarrow (w_2 \vee x_1)) \wedge w_3$

replace with 3-CNF Equivalent

x_1	x_2	w_1	$(x_1 \wedge x_2)$	$\neg(w_1 \leftrightarrow (x_1 \wedge x_2))$	
0	0	0	0	0	
0	0	1	0	1	← $\bar{x}_1 \bar{x}_2 w_1$
0	1	0	0	0	
0	1	1	0	1	← $\bar{x}_1 x_2 w_1$
1	0	0	0	0	
1	0	1	0	1	← $x_1 \bar{x}_2 w_1$
1	1	0	1	1	← $x_1 x_2 \bar{w}_1$
1	1	1	1	0	

$(x_1 \vee x_2 \vee \bar{w}_1) \wedge (x_1 \vee \bar{x}_2 \vee \bar{w}_1) \wedge (\bar{x}_1 \vee x_2 \vee \bar{w}_1) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee w_1)$

How do you prove problem *A* is NP-complete?

1) **Prove *A* is in NP:** show that given a solution, it can be verified in polynomial time.

2) **Prove that *A* is NP-hard:**

   a) Select a **known** NP-complete problem *B*.

   b) Describe a polynomial time computable algorithm that computes a function *f*, **mapping *every* instance of *B* to an instance of *A***. (that is: $B \le_p A$)

   c) Prove that every ***yes***-instance of *B* maps to a ***yes***-instance of *A*, and every ***no***-instance of *B* maps to a ***no***-instance of *A*.

   d) Prove that the algorithm computing ***f* runs in polynomial time**.

Proof that problem *A* is NP-complete

1) **Prove *A* is in NP: "**Given a possible solution to *A*, I can verify its correctness in polynomial-time."

2) **Prove that *A* is NP-hard:**

   a) "I will reduce known NP-complete problem *B* to *A.*"

   your function f

   b) "Let ***b*** be an arbitrary instance of problem *B*. Here is how you convert ***b*** to an instance ***a*** of problem *A*." Note: this method must work for ANY instance of *B*.

   c) "If ***a*** is a "yes"-instance, then this implies that ***b*** is also a "yes"-instance. **Conversely**, if ***b*** is a "yes"-instance, then this implies that ***a*** is also a "yes"-instance."

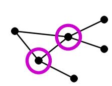
   d) "The conversion from *B* to *A* runs in polynomial time because…."

NP-complete problem: **Vertex Cover**

Input: Undirected graph $G = (V, E)$, integer k.

Output: True iff there is a subset C of V of size $\le k$ such that every edge in E is incident to at least one vertex in C.

Example: Vertex cover of size ≤ 2.

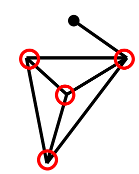


NP-complete problem: **Clique**

Input: Undirected graph $G = (V, E)$, integer k.

Output: True iff there is a subset C of V of size $\ge k$ such that all vertices in C are connected to all other vertices in C.

Example: Clique of size ≥ 4



NP-complete problem: **Satisfiability (SAT)**

Input: A Boolean formula in CNF form.

Output: True iff there is a truth assignment of 0's and 1's to the variables such that the value of the expression is 1.

Example: Formula *S* is satisfiable with the truth assignment $x=1, y=1$ and $z=0$.

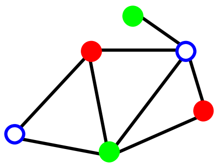
$$S=(x+y+\neg z)\bullet(\neg x+y+z)\bullet(\neg x+\neg y+\neg z)$$

NP-complete problem: **3-Coloring**

Input: An undirected graph G=(V,E).

Output: True iff there is an assignment of colors to the vertices in G such that no two adjacent vertices have the same color. (using only 3 colors)

Example:



NP-complete problem: **Knapsack**

Input: set of objects with weights and values, a maximum weight that can be carried and a desired value. (see p. 357 in Manber)

Output: True iff there is a subset of the objects with (total weight ≤ *allowable weight)* and (total value ≥ *desired value)*.

Example: Items: {a, b, c},size(a)=3,size(b)=6,size(c)=4

value(a)=$30, value(b)=$24, value(c)=$18

Max weight = 10, Desired value = $50.

Answer: yes, {a,b}

NP-complete problem: **Partition**

Input: Set of items *S*, each with an associated size. The sum of the items' sizes is *2k*.

Output: True iff there is a subset of the items whose sizes add up to *k*.

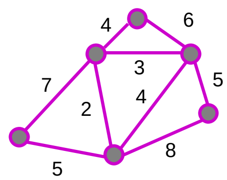
Example: *S* = (2,3,1,10,4,6). Is there a subset of items that sums to 13? (yes)

NP-complete problem: **TSP**

Input: An undirected graph G=(V,E) with integer edge weights, and an integer b.

Output: True iff there is a simple cycle in G passing through all vertices (once), with total cost ≤ b.

Example:
b = 34



$3SAT \le_m^P$ Vertex Cover

Def: VertexCover(VC) = { <G, K> | undirected graph G has a vertex cover of size ≤ K }. $U \subseteq V$ is a vertex cover if all edges touch some $u \in U$.

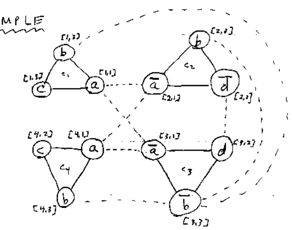
3SAT Instance:
Variables: a, b, c, ...
literals: $\alpha_{i,j}$, $1 \le i \le q$, $1 \le j \le 3$
Clauses: $c_i = \alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$, $1 \le i \le q$
Formula: $F = c_1 \wedge c_2 \cdots \wedge c_q$

$\xrightarrow{f}$

VC Instance:
$K = 2q$
$G = (V, E)$
$V = \{ [i,j] \mid 1 \le i \le q, 1 \le j \le 3 \}$
$E = \{ ([i,j],[k,l]) \mid i = k$, or $\alpha_{ij} = \neg \alpha_{kl} \}$

EXAMPLE

c_1 $(a \vee b \vee c)$
c_2 $\wedge (\bar{a} \vee \bar{d} \vee b)$
c_3 $\wedge (\bar{a} \vee d \vee \bar{b})$
c_4 $\wedge (a \vee c \vee b)$



P-time: Easy

Correctness: $w \in 3SAT \Leftrightarrow f(w) \in VC$

① (⇒) Let $\vec{x}$ be a satisfying assignment

1a Choose $U \subseteq V$, $|U| = k = 2q$:
For $1 \le i \le q$, let j_i be s.t. α_{i,j_i} = true,
$U = \{ [i,j] \mid j \ne j_i \}$

1b Prove U is a vertex cover:
(i) includes 2 of 3 vertices in each △ so those edges covered
(ii) only true literals omitted from U, so for every ⓐ - - ⓐ̄ edge, at least one end in U.

② (⇐) Let $U \subseteq V$ be a vertex cover, $|U| \le 2q$.

2a Choose a set of literals:
$\mathcal{A} = \{ \alpha_{i,j} \mid [i,j] \notin U \}$

2b Prove $\mathcal{A}$ = TRUE is a satisfying assignment
(i) $\mathcal{A}$ contains no pair of complementary literals $\alpha_{ij} = \neg\alpha_{kl}$, else [i,j] - - - [k,l] uncovered. ∴ $\mathcal{A}$ is an assignment.
(ii) Since $|U| \le 2q$, if any △ has all 3 vertices in U, then some other has ≤1 vertex in U, contradicting "U is a v.c."
∴ $|U| = 2q$, & each triangle has exactly 2 in U, ∴ each clause has a true literal

$3SAT \le_m^P$ 3COLOR

Def 3COLOR = { G | G is a 3-colorable graph }

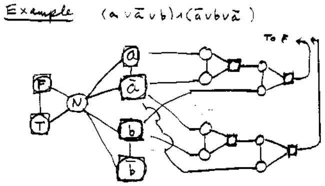
G = (V, E) is 3-colorable iff there is a function $\chi : V \to \{1, 2, 3\}$ st $\chi(u) \ne \chi(v)$ For all $(u,v) \in E$.

3SAT instance:
Variables: $v_1, v_2, \ldots, v_n$
literals: $\alpha_{i,j}$, $1 \le i \le q$, $1 \le j \le 3$
Clauses: $C_i = \alpha_{i1} \vee \alpha_{i2} \vee \alpha_{i3}$, $1 \le i \le q$
Formula: $F = c_1 \wedge c_2 \wedge \cdots \wedge c_q$

$\xrightarrow{f}$

3COLOR instance:
$6q + 2n + 3$ vertices
$13q + 3n + 3$ edges
(See Example)

Example $(a \vee \bar{a} \vee b) \wedge (\bar{a} \vee b \vee \bar{a})$



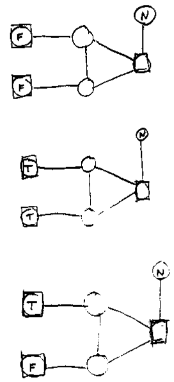
All square nodes connect to N

A 3-Coloring Gadget

"Sort of an OR gate":

(1) if any input is T,
 the output can be T

(2) if output is T,
 some input must be T

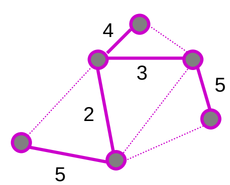


Coping with NP-Completeness

- Is your real problem a special subcase?
  - E.g. 3-SAT is NP-complete, but 2-SAT is not;
  - Ditto 3- vs 2-coloring
  - E.g. maybe you only need planar graphs, or degree 3 graphs, or …
- Guaranteed approximation good enough?
  - E.g. Euclidean TSP within 1.5 * Opt in poly time
- Clever exhaustive search, e.g. Branch & Bound
- Heuristics – usually a good approximation and/or usually fast

2x Approximation to EuclideanTSP

- A TSP tour visits all vertices, so contains a spanning tree, so TSP cost is > cost of min spanning tree.
- Find MST
- Double all edges
- Find Euler Tour
- Shortcut
- Cost of shortcut < ET = 2 * MST < 2 * TSP



1.5x Approximation to EuclideanTSP

- Find MST
- Find min cost matching among odd-degree tree vertices
- Cost of matching ≤ TSP/2
- Find Euler Tour
- Shortcut
- Shortcut ≤ ET ≤ MST + TSP/2 < 1.5* TSP