

Minimum Cost Spanning Trees

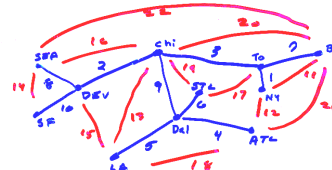
Kruskal's Algorithm:

Another Example of the Greedy Method

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Min Cost Spanning Tree

- Given: Undirected graph $G = (V, E)$ & positive edge cost/weight $c(e) \in \mathbb{R}$ for each $e \in E$.
- Find: connected $T \subseteq E$ minimizing $\sum_{e \in T} c(e)$



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Applications

- Broadcast tree in a network
- Building roads or power lines
- Routing power & ground on a PC board
- Clustering
- ...

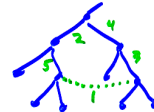
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Lemma 1: Trees and Cycles

Adding an edge to a tree creates a cycle;
deleting any cycle edge gives a tree

- Corollary 1: Solution to MST is a tree
- Corollary 2: Cheapest edge in E is in T
- Exercises:
 - 2nd-cheapest edge also in T
 - 3rd-cheapest?



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Kruskal's Algorithm

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sort edges
K := ∅
while |K| < n-1 do
  e := next cheapest edge
  if K ∪ {e} is acyclic
    then K := K ∪ {e}
  else discard e
    
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Correctness

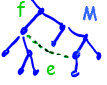
Theorem: Kruskal's algorithm builds an MST
Proof:

- Suppose Kruskal picks the tree K
- Suppose MST M maximizes $|K \cap M|$ among all MSTs
- For sake of contradiction, suppose $K \neq M$
- Let e be the cheapest edge in $K - M$
- Then...

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Claim



$M \cup \{e\}$ has a cycle containing an edge f s.t.

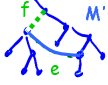
- (1) $f \notin K$, and
- (2) $c(f) \geq c(e)$

Proof:

- (1) If all the cycle edges were in K , then K wouldn't be a tree.
- (2) If $c(f) < c(e)$, **greedy looked at f before e** .
But $\{e' \in K \mid c(e') < c(e)\} \cup \{f\} \subseteq M$,
hence **acyclic**, so f would have been picked, but it wasn't.

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Correctness (cont.)



Theorem: Kruskal's algorithm builds an MST

Proof:

- Suppose Kruskal picks the tree K
- Suppose MST M maximizes $|K \cap M|$ among all MSTs
- For sake of contradiction, suppose $K \neq M$
- Let e be the cheapest edge in $K - M$
- From claim, $\exists f$ in cycle s.t. $f \notin K$, $c(f) \geq c(e)$
- Let $M' = (M \cup \{e\}) - \{f\}$
- Then M' is an MST with $|K \cap M'| > |K \cap M|$.
Contradiction. QED

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Implementation

Testing "if $K \cup \{e\}$ is acyclic":

Union/Find problem

- Linear space
- Time $n \alpha(n)$
- $\alpha(n) < 5$ for all $n < \text{age of universe}$

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