CSE 421: Introduction to Algorithms

Dynamic Programming

"Dynamic Programming"

Program — A plan or procedure for dealing with some matter – Webster's New World Dictionary

Dynamic Programming

- Outline:
  - Example 1 – Licking Stamps
  - General Principles
  - Example 2 – Knapsack (§ 5.10)
  - Example 3 – Sequence Comparison (§ 6.8)

Licking Stamps

- Given:
  - Large supply of 5¢, 4¢, and 1¢ stamps
  - An amount N
- Problem: choose fewest stamps totaling N

How to Lick 27¢

# of 5¢ Stamps	# of 4¢ Stamps	# of 1¢ Stamps	Total Number
5	0	2	7
4	1	3	8
3	3	0	6

Moral: Greed doesn't pay

A Simple Algorithm

- At most N stamps needed, etc.

```
for a = 0, …, N {
  for b = 0, …, N {
    for c = 0, …, N {
      if (5a+4b+c == N && a+b+c is new min)
        {retain (a,b,c);}}}
output retained triple;
```

- Time: $O(N^3)$
  (Not too hard to see some optimizations, but we're after bigger fish…)

Better Idea

Theorem: If last stamp licked in an optimal solution has value v, then previous stamps form an optimal solution for N-v.

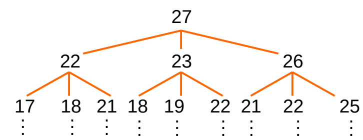
Proof: if not, we could improve the solution for N by using opt for N-v.

$$M(i) = \min \begin{Bmatrix} 0 & i=0 \\ 1+M(i-5) & i\ge 5 \\ 1+M(i-4) & i\ge 4 \\ 1+M(i-1) & i\ge 1 \end{Bmatrix}$$

where $M(i)$ = min number of stamps totaling i¢

New Idea: Recursion

$$M(i) = \min \begin{Bmatrix} 0 & i=0 \\ 1+M(i-5) & i\ge 5 \\ 1+M(i-4) & i\ge 4 \\ 1+M(i-1) & i\ge 1 \end{Bmatrix}$$



Time: $> 3^{N/5}$

Another New Idea: Avoid Recomputation

- Tabulate values of solved subproblems
  - Top-down: "memoization"
  - Bottom up:

for i = 0, …, N do $M(i) = \min \begin{Bmatrix} 0 & i=0 \\ 1+M(i-5) & i\ge 5 \\ 1+M(i-4) & i\ge 4 \\ 1+M(i-1) & i\ge 1 \end{Bmatrix};$

- Time: O(N)

Finding How Many Stamps

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

1+Min(3,1,3) = 2

Finding Which Stamps: Trace-Back

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
M(i)	0	1	2	3	1	1	2	3	2						

4¢

<u>1</u>+Min(3,<u>1</u>,3) = <u>2</u>

Complexity Note

- O(N) is better than O(N^3) or O($3^{N/5}$)
- But still *exponential* in input size (log N bits)

  (E.g., miserably slow if N is 64 bits – $c \cdot 2^{64}$ steps for 64 bit input.)
- Note: can do in O(1) for 5¢, 4¢, and 1¢ but not in general. See "NP-Completeness" later

Elements of Dynamic Programming

- What feature did we use?
- What should we look for to use again?

- "Optimal Substructure"
  - Optimal solution contains optimal subproblems
- "Repeated Subproblems"
  - The same subproblems arise in various ways

The Knapsack Problem (§ 5.10)

Given positive integers $W, w_1, w_2, \ldots, w_n,$

Find a subset of the w_i's totaling exactly W.

Alternate (Easier?) Problem: Is there one?

(Like stamp problem, but limited supply of each.)

Motivation: simple 1-d abstraction of packing boxes, trucks, VLSI chips, …

Knapsack Example

$w_1, \ldots, w_4 = 2, 5, 9, 11$

- $W = 14$
  - YES: $5+9 = 14$
- $W = 15$
  - NO:
    - all singletons ≤ 11: too small
    - all pairs too small, except 9+11, 5+11 too big
    - all triples ≥ 16: too big
    - all quadruples: too big

  2^n possibilities

Solve by Induction? Try 1

- Defn: Let $P(i)$ be true iff there is a subset of first i weights $w_1, w_2, \ldots, w_i$ totaling W
- Assume we know how to evaluate $P(n-1)$
  - Case 1: $P(n-1)$ = True – done; w_n unneeded
  - Case 2: $P(n-1)$ = False – may or may not be a solution, but if there is one, it *in*cludes w_n, and other included weights total $W-w_n$, but I.H. doesn't tell us how to find it. ☹

Solve by Induction? Try 2

- Defn: Let $P(i, X)$ be true iff there is a subset of first i weights $w_1, w_2, \ldots, w_i$ totaling X
- Assume we know $P(n-1, X)$ for all $X \le W$
  - Case 1: $P(n-1, W)$ = True – done; w_n unneeded
  - Case 2: $P(n-1, W)$ = False – may or may not be a solution, but if there is one, it *in*cludes w_n, and other weights total $W-w_n$, so $P(n, W) = P(n-1, W-w_n)$ ☺
- Algorithm:
  - $P(n,W) = P(n-1, W) \vee P(n-1, W-w_n)$
  - Basis: $P(0, X)$ = True iff $(X == 0)$

Knapsack Example

$\boxed{P(n,W) = P(n-1, W) \vee P(n-1, W-w_n)}$

$w_1, \ldots, w_4 = 2, 5, 9, 11$ W=15

i\X	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0
3	1	0	1	0	0	1	0	1	0	1	0	1	0	0	1	0
4	1	0	1	0	0	1	0	1	0	1	0	1	0	0	1	0

W = 14: Yes

W = 15: No

Dynamic Programming?

$P(n,W) = P(n-1, W) \vee P(n-1, W-w_n)$

- Optimal substructure?
  Best/only way to fill a big knapsack implicitly fills smaller ones with fewer objects in the best or only way
- Repeated subproblems?
  Smallest cases potentially common to many bigger instances

Complexity Notes

- Time is O(N W)
- May or may not beat naïve 2^N
- But still partially *exponential* in input size (N log W bits)
  - E.g., 100 weights, 64 bits each – 100 • 2^{64} array elements.
  - C.v., e.g., Skyline 100 bldgs, 64 bit coords – c • 100 • log 100 steps.
- See "NP-Completeness" later