Dynamic Programming on Graphs

CSE 417 Winter 24
Lecture 15

New Resource on Assignments Page

For a few homeworks a “sample problem” with a “sample solution”

Not the actual problems on the homework, but a problem covering a similar topic (e.g., a divide and conquer question, a dynamic programming question, etc.).

Since we don’t have staff solutions (so you can resubmit!) we wanted to give you something to refer to for ‘style’/length/etc.

Lecture examples are also good! But it’s sometimes nice to see a full answer all in one spot.

Dynamic Programming

So far: DP on arrays and lists.

The end of yesterday's slide deck has practice materials:

- Do LIS again, but from "left-to-right" instead of "right-to-left" (try doing the same problem again to see if you understood it).

- Another problem (subset sum)

Today: our last DP day---DP on trees and graphs

DP on trees is very common; we'll practice it today. DP on graphs is less common; we'll give you intuition why and tell you about a famous one.

Lots of slides we're intentionally skipping

DP on Trees

Trees are recursive structures

A tree is a root node, with zero or more children

Each of which are roots of trees

Since DP is "smart recursion" (recursion where we save values)

Recursive functions/calculations are really common.

DP on Trees

Find the minimum vertex cover in a tree.

Give every **vertex** a weight, find the minimum weight vertex cover

Vertex Cover

A set S of vertices is a vertex cover if for every edge (u, v):
u is in S, or v is in S, (or both)

The weight of a vertex cover is just the sum of the weights of the vertices in the set.

We want to find the minimum weight vertex cover.

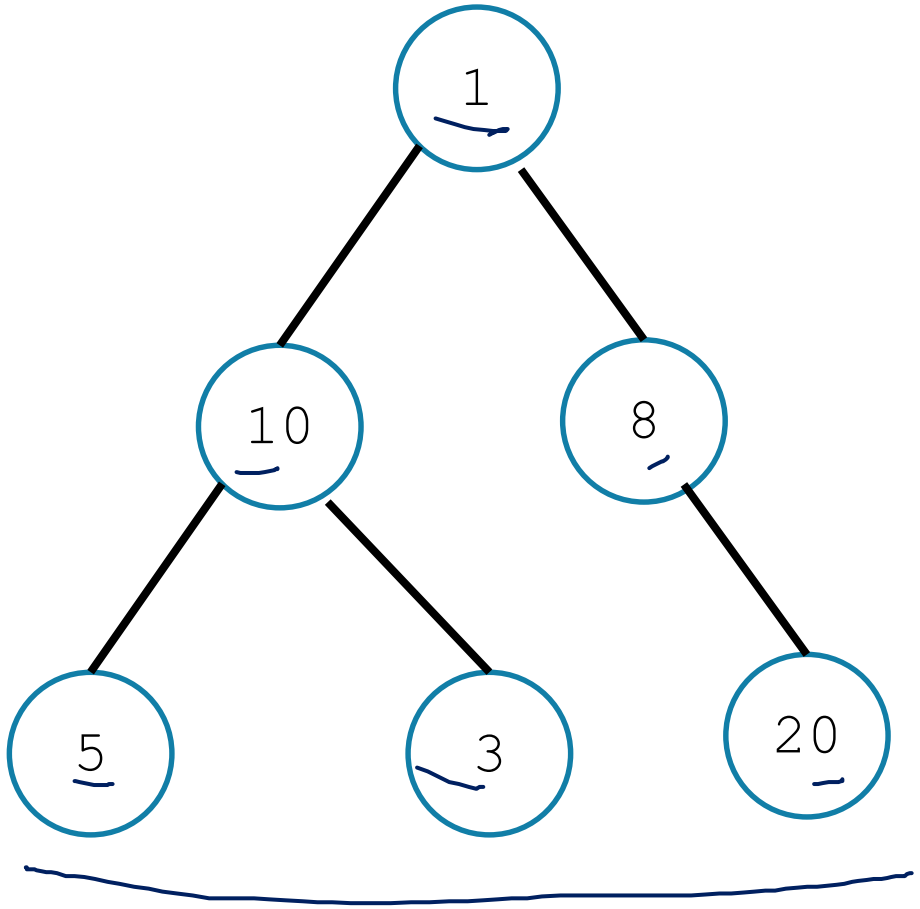
Vertex Cover

Vertex Cover

A set S of vertices is a vertex cover if for every edge (u, v): u is in S, or v is in S, (or both)

Find the minimum vertex cover in a tree.

Give every **vertex** a weight, find the minimum weight vertex cover



Vertex Cover

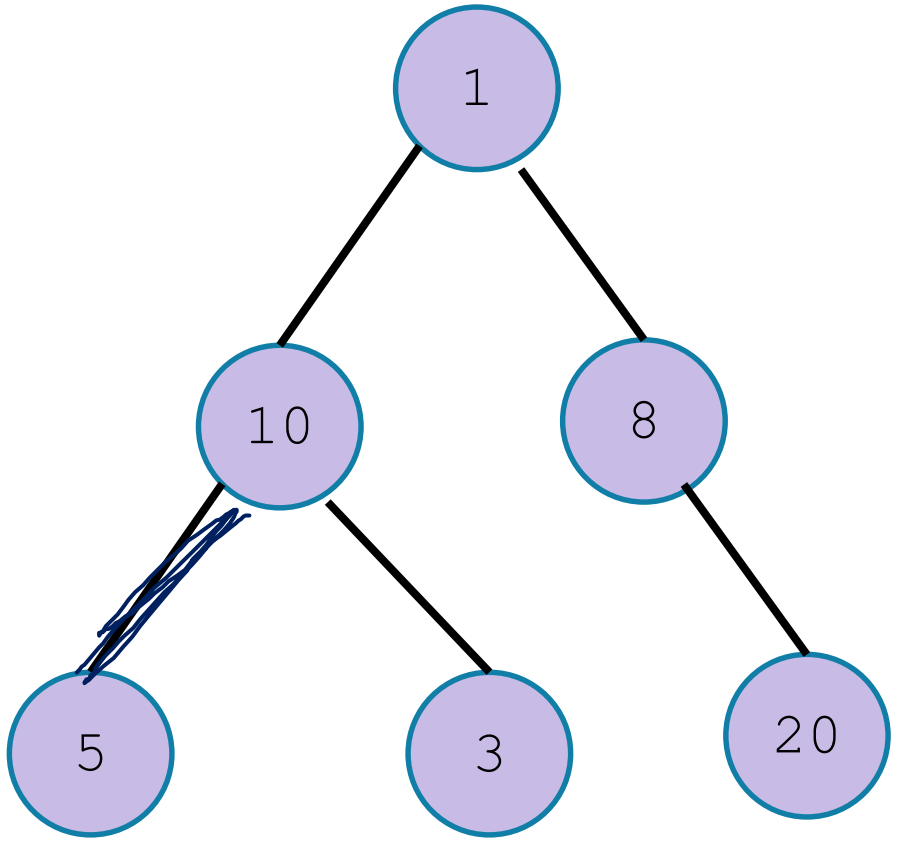
Vertex Cover

A set S of vertices is a vertex cover if for every edge (u, v): u is in S, or v is in S, (or both)

Find the minimum vertex cover in a tree.

Give every **vertex** a weight, find the minimum weight vertex cover

A valid vertex cover! (just take everything)
Definitely not the minimum though.



Vertex Cover

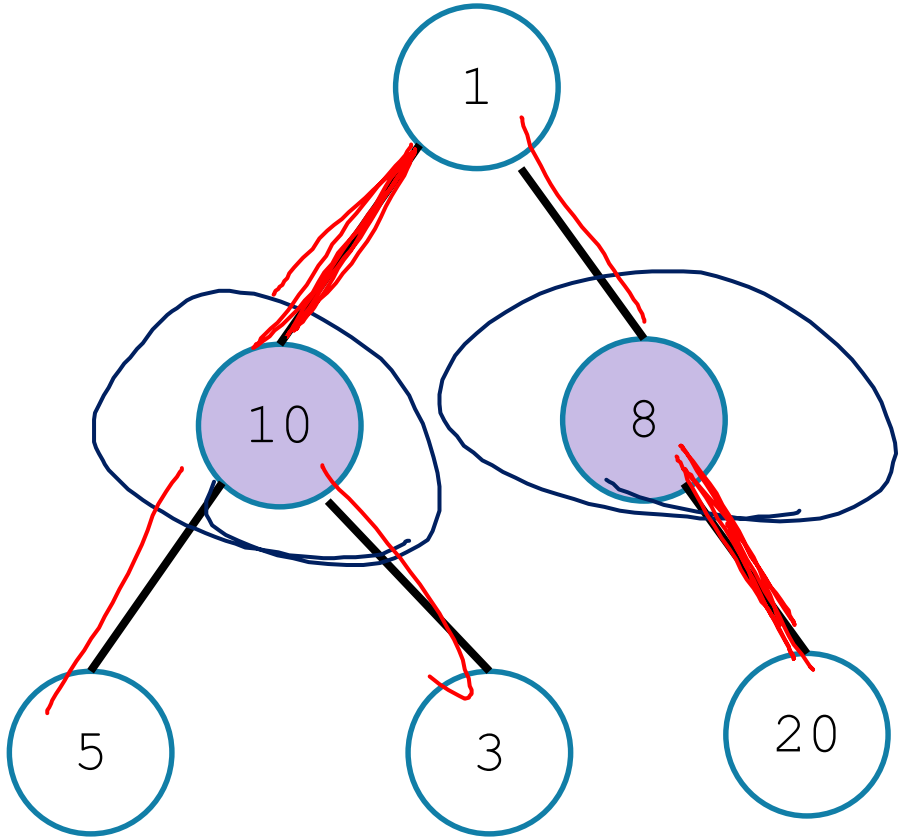
Vertex Cover

A set S of vertices is a vertex cover if for every edge (u, v): u is in S, or v is in S, (or both)

Find the minimum vertex cover in a tree.

Give every **vertex** a weight, find the minimum weight vertex cover

A better vertex cover – weight 18



Vertex Cover

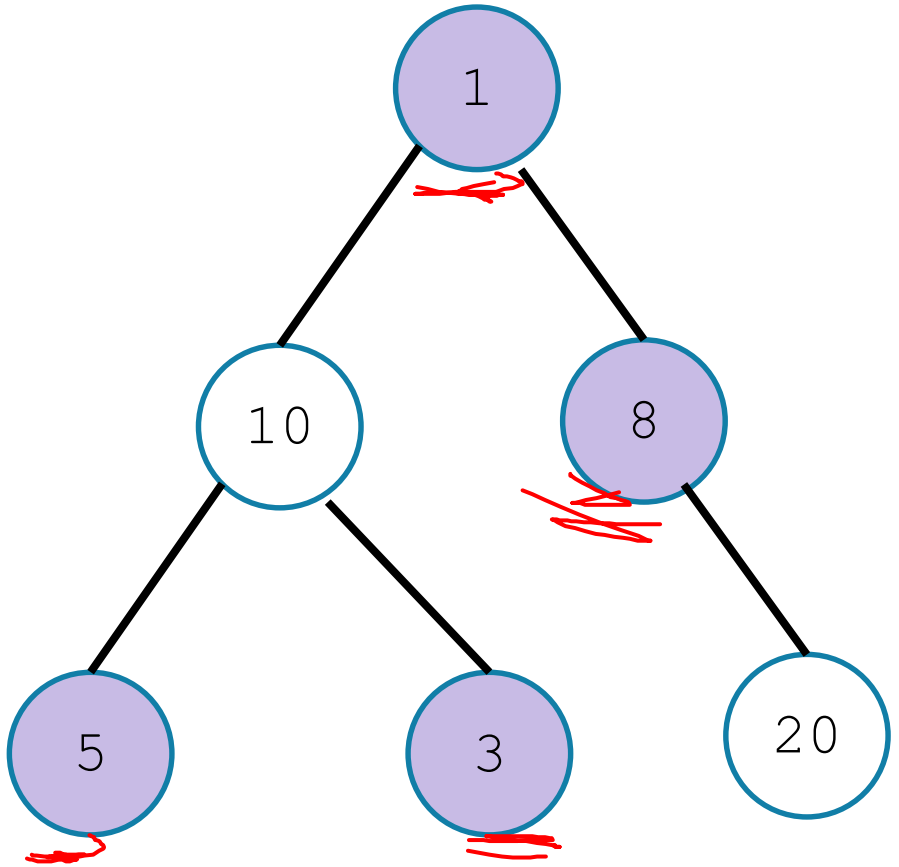
Vertex Cover

A set S of vertices is a vertex cover if for every edge (u, v): u is in S, or v is in S, (or both)

Find the minimum vertex cover in a tree.

Give every **vertex** a weight, find the minimum weight vertex cover

The minimum vertex cover: weight 17



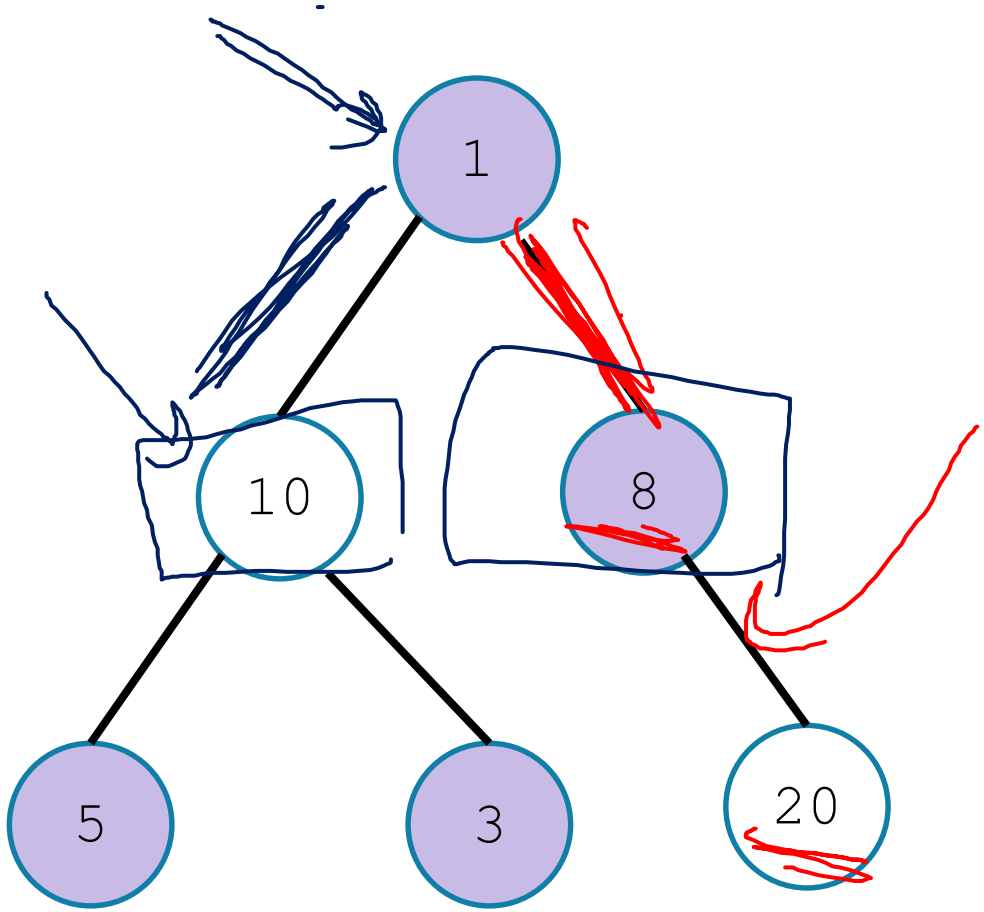
Vertex Cover

Vertex Cover

A set S of vertices is a vertex cover if for every edge (u, v): u is in S, or v is in S, (or both)

Notice, the minimum weight vertex cover might have both endpoints of some edges

Even though only one of 1, 8 is required on the edge between them, they are both required for other edges.



Step 1: Formulate your problem recursively

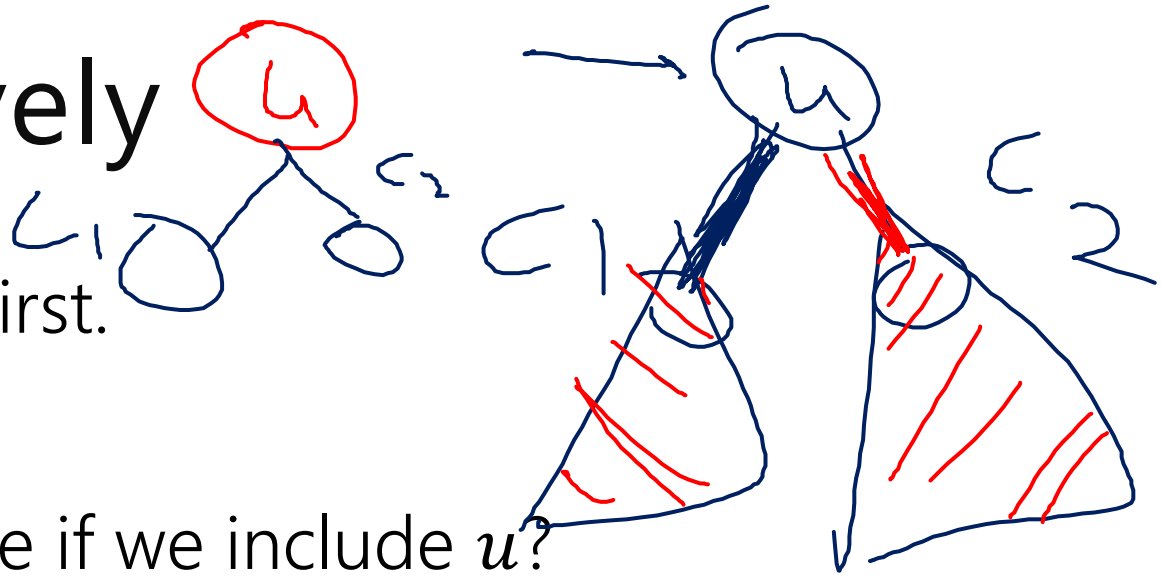
Try top down and bottom up thinking.

Top down: For one particular "element" of the input, what do I need to decide? What are all the possibilities?

Bottom up: What is a smaller version of the problem?

What information do we need to solve the problem recursively?

Vertex Cover – Recursively



Let's try to write a recursive algorithm first.

What information do we need to decide if we include u?

If we don't include u then to be a valid vertex cover we need…

If we do include u then to be a valid vertex cover we need…

Vertex Cover – Recursively

Let's try to write a recursive algorithm first.

What information do we need to decide if we include u?

If we don't include u then to be a valid vertex cover we need…

to include **all** of $u's$ children, and vertex covers for each subtree

If we do include u then to be a valid vertex cover we need…

just vertex covers in each subtree (whether children included or not)

Step 2: What do you need recursively

Write **in English** (not math. English.) what exactly the recursive call is giving you.

I know this sounds silly. Every time I am trying to solve a DP problem from scratch (or just remember how one worked) and can't do it, it's because I haven't written it down in English.

If you realize "I need an extra parameter" update the English description with that extra parameter.

Recurrence

Let $OPT(v)$ be the weight of a minimum weight vertex cover for the subtree rooted at v.

For this recurrence we are ignoring any edge from v to a parent (if one exists).

Write a recurrence for $OPT()$

Then figure out how to calculate it.

Hint: you need to change or add something.

Recurrence

$OPT(v)$ – the weight of the minimum weight vertex cover for the tree rooted at v (whether or not v is included).

$INCLUDE(v)$ – the weight of the minimum weight vertex cover for the tree rooted at v where v is included in the vertex cover.

$$OPT(v) = \begin{cases} \min\{\sum_{u:u \text{ is a child of } v} INCLUDE(u), weight(v) + \sum_{u:u \text{ is a child of } v} OPT(u)\} & \text{if } v \text{ is not a leaf} \\ 0 \text{ if } v \text{ is a leaf} \end{cases}$$

$$INCLUDE(v) = weight(v) + \sum_{u:u \text{ is a child of } v} OPT(u)$$

Vertex Cover Dynamic Program

What memoization structure should we use?

What code should we write?

What's the running time?

Vertex Cover Dynamic Program

What memoization structure should we use?

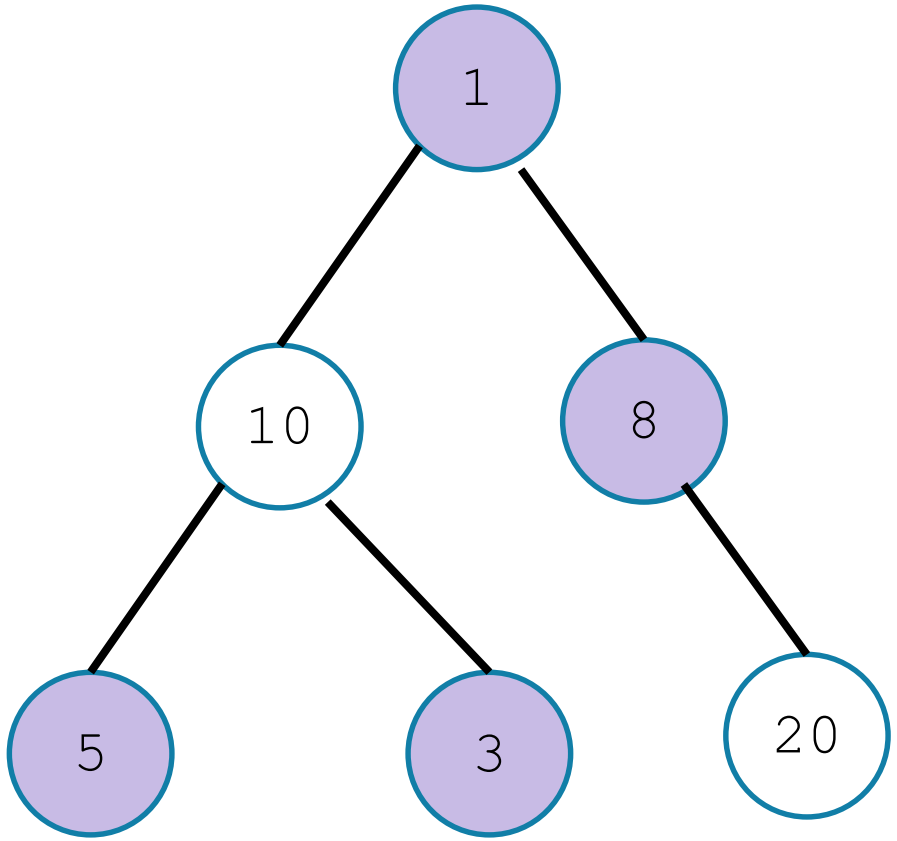
the tree itself!

What code should we write?

What's the running time?

Vertex Cover

What order do we do the calculation?



Vertex Cover Dynamic Program

What memoization structure should we use?

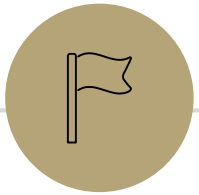
the tree itself!

What code should we write?

A post-order traversal (make recursive calls, then look up values in children to do calculations)

What's the running time?

$\Theta(n)$



DP on graphs

Rest of this slide deck

Dynamic Programming on Graphs

We're building up to "Bellman-Ford" and "Floyd-Warshall"

Two very clever algorithms – we won't ask you to be as clever.

But they're standard library functions, so it's good to know.

And deriving them together is good for practicing DP skills.

Want to understand: why is DP on graphs with cycles harder than DP on trees?

Shortest Paths

Shortest Path Problem

Given: A directed graph and a vertex s

Find: The length of the shortest path from s to t.

The length of a path is the sum of the edge weights.

Baseline: Dijkstra's Algorithm

Dijkstra's Algorithm

```
Dijkstra(Graph G, Vertex source)
     initialize distances to ∞
      mark source as distance 0
      mark all vertices unprocessed
      while(there are unprocessed vertices){
            let u be the closest unprocessed vertex
            foreach(edge (u,v) leaving u){
                  if(u.dist+weight(u,v) < v.dist){
                        v.dist = u.dist+weight(u,v)
                        v.predecessor = u
                  }
            }
       mark u as processed
      }
```

In 373, we said the running time was $O(m \log n + n \log n)$

Can be sped up to $O(m + n \log n)$ by inserting a different heap implementation.

A recurrence

Suppose you have a directed acyclic graph G.

How could you find distances from s?

What's one step in this problem?

A recurrence

Suppose you have a directed acyclic graph G.

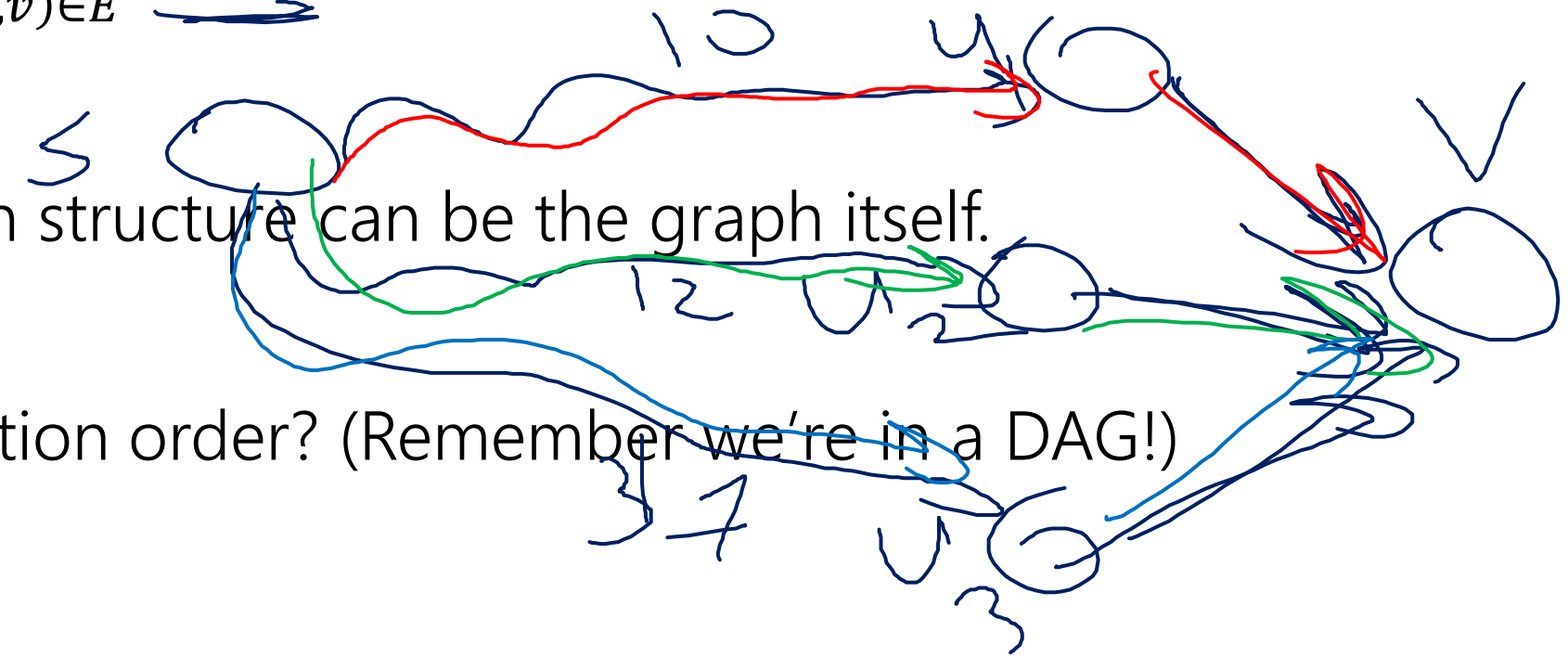
How could you find distances from s?

What's one step in this problem?

Choosing the predecessor, i.e. "the last edge" on a path.

A recurrence

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$



Our memoization structure can be the graph itself.

What's an evaluation order? (Remember we're in a DAG!)

A recurrence

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$

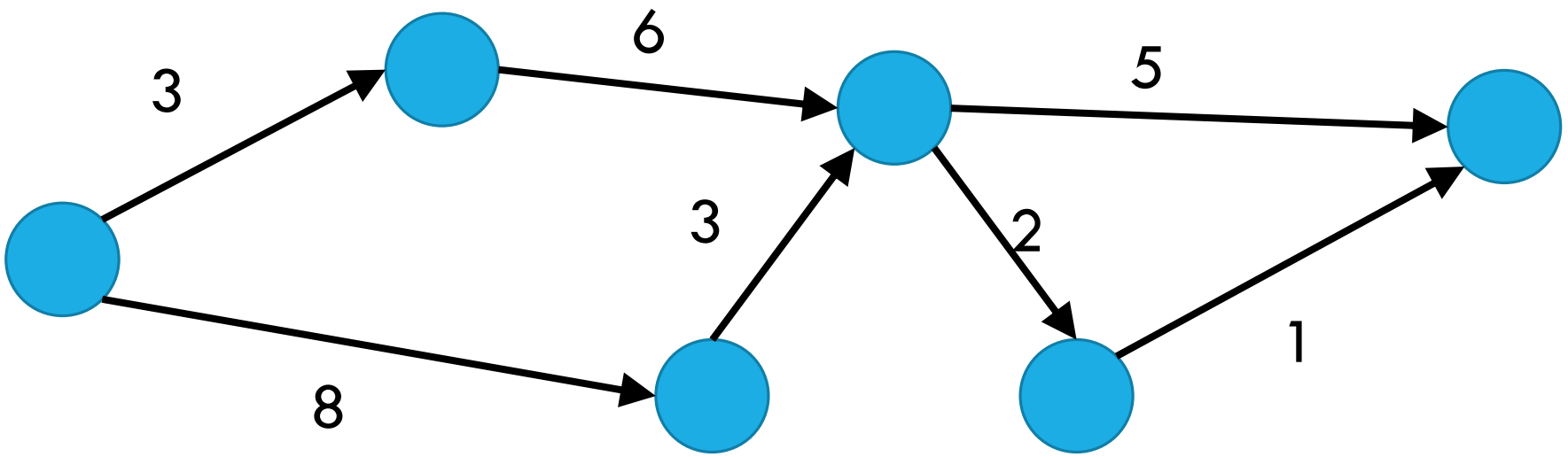
Our memoization structure can be the graph itself.

What's an evaluation order? (Remember we're in a DAG!)

A topological sort! – we need to have distances for all incoming edges calculated.

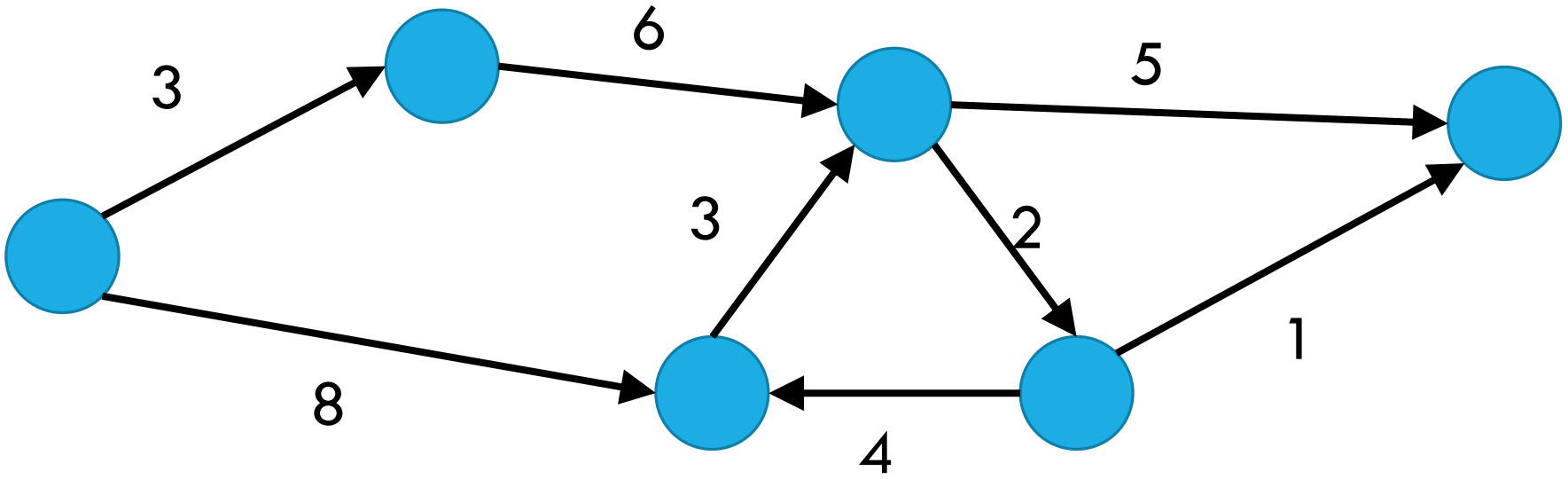
In a DAG

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$



What about cycles?

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$



Cycles

We need some way to “order” the paths.

I.e. we need to be sure we always have **something** to look up.

It doesn’t have to be the perfect distance necessarily…

As long as we’ll realize it and update later

And as long as we can fix it to the true distance eventually.

Ordering

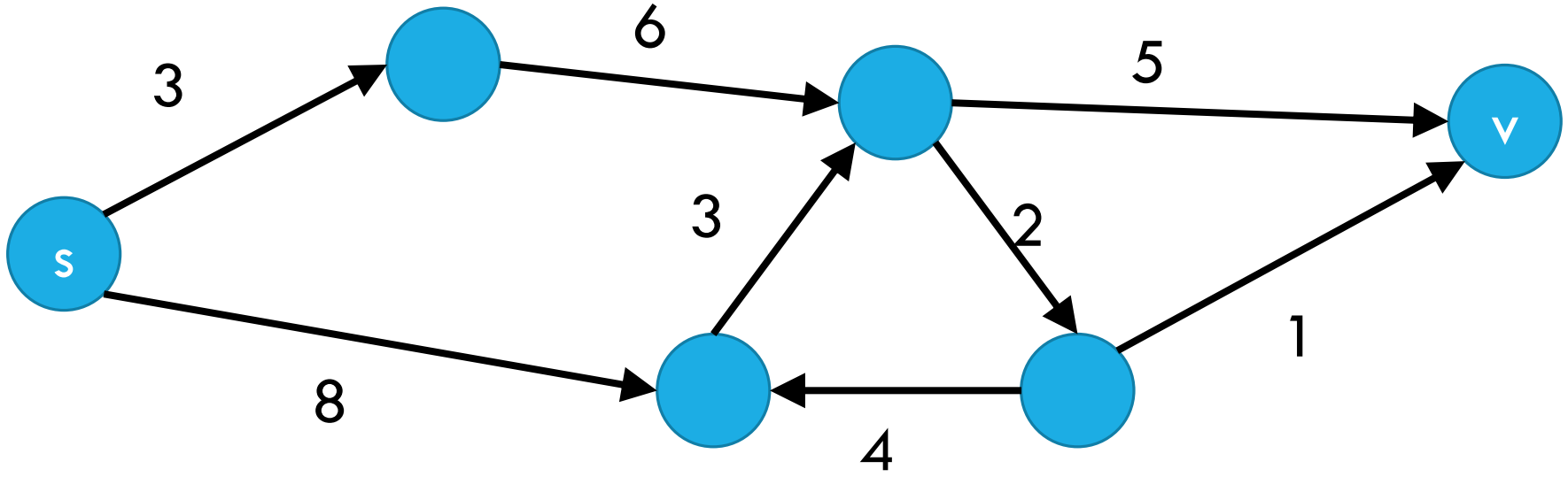
Instead of $dist(v)$, (the true distance) right from the start, we'll let

$dist(v, i)$ to be the length of the shortest path from the source to v that uses at most i edges.

That breaks ties – counting the number of edges required!

$dist(v, i) =$

Distances



$dist(v, 2) = \infty$ (can't get there in 2 hops)

$dist(v, 3) = 14$

$dist(v, 4) = 12$

Ordering

Instead of $dist(v)$, (the true distance) right from the start, we'll let

$dist(v, i)$ to be the length of the shortest path from the source to v that uses at most i edges.

That breaks ties – counting the number of edges required!

$dist(v, i) =$

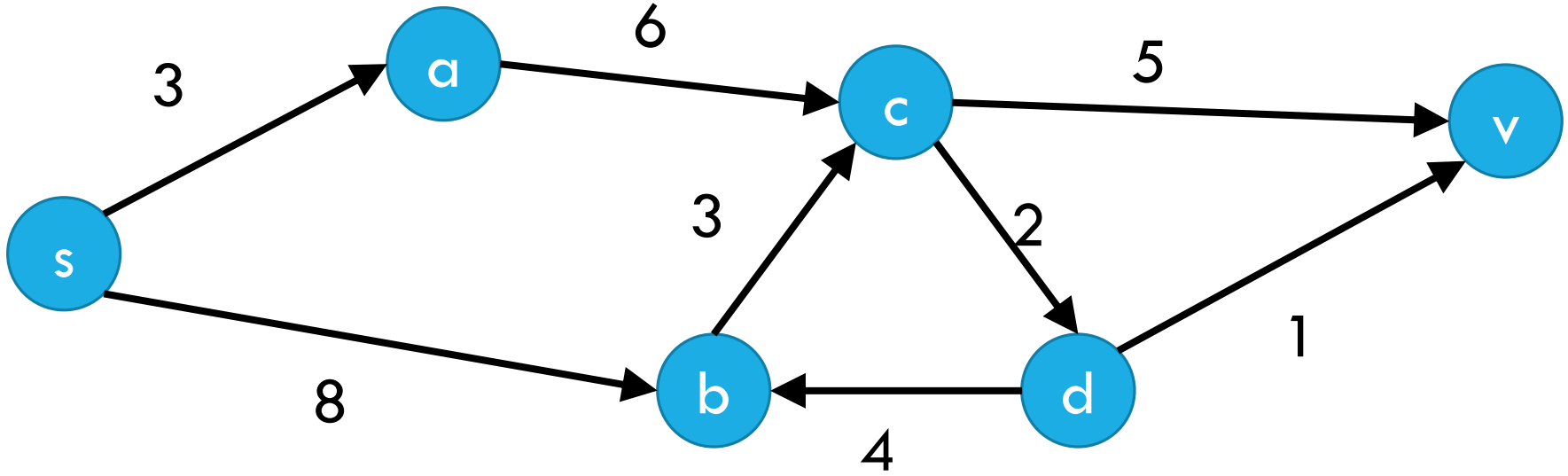
Ordering

Instead of $dist(v)$, we want the

$dist(v, i)$ to be the length of the shortest path from the source to u that uses at most i edges.

$$dist(v,i) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v \text{ is the source} \\ \infty & \text{if } i = 0 \text{ and } v \text{ is not the source} \\ \min\left\{ \min_{u:(u,v)\in E} \{dist(u, i-1)\} + w(u,v), dist(v, i-1) \right\} & \text{o/w} \end{cases}$$

Sample calculation



Vertex\i	0	1	2	3	4	5
S	0	0	0	0	0	0
A	∞	3	3	3	3	3
B	∞	8	8	8	8	8
C	∞	∞	9	9	9	9
D	∞	∞	∞	11	11	11
V	∞	∞	∞	14	12	12

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to ??)
    for(every vertex v) //what order?
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

$$dist(v,i) = \begin{cases} 0 & \text{if } i = 0 \text{ and } v \text{ is the source} \\ \infty & \text{if } i = 0 \text{ and } v \text{ is not the source} \\ \min\left\{ \min_{u:(u,v)\in E} \{dist(u,i-1)\} + w(u,v), dist(v,i-1) \right\} \end{cases}$$

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    for(every vertex
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

The shortest path will never need more than $n-1$ edges (more than that and you've got a cycle)

Pseudocode

```
Initialize sour
for(i from 1 to
   for(every vertex v) //what order?
      v.dist[i] = v.dist[i-1]
       for(each incoming edge (u,v))//hmmm
          if(u.dist[i-1]+weight(u,v)<v.dist[i])
             v.dist[i]=u.dist[i-1]+weight(u,v)
          endIf
       endFor
   endFor
endFor
```

Only ever need values from the last iteration
Order doesn't matter!!

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    for(every vertex v) //any order
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

Graphs don't usually have easy access to their incoming edges (just the outgoing ones)

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    for(every vertex v) //any order
        v.dist[i] = v.dist[i-1]
        for(each incoming edge (u,v))//hmmm
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

But the order doesn't matter – as long as we check every edge, the processing order is irrelevant.
So if we only have access to outgoing edges…

Pseudocode

```
Initialize source.dist[0]=0, u.dist[0]=∞ for others
for(i from 1 to n-1)
    set u.dist[i] to u.dist[i-1] for every u
    for(every vertex u) //any order
        for(each outgoing edge (u,v))//better!
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

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for(i from 1 to n-1)
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        for(each outgoing edge (u,v))//better!
            if(u.dist[i-1]+weight(u,v)<v.dist[i])
                v.dist[i]=u.dist[i-1]+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

We don't really need all the different values…
Just the most recent value.

Pseudocode

```
Initialize source.dist=0, u.dist=∞ for others
for(i from 1 to n-1)
    set u.dist[i] to u.dist[i-1] for every u
    for(every vertex u) //any order
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endFor
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                v.dist=u.dist+weight(u,v)
            endIf
        endFor
    endFor
endFor
```

We don't really need all the different values…
Just the most recent value.

A Caution

We did change the code when we got rid of the indexing

You might have a mix of `dist[i]`,`dist[i+1]`,`dist[i+2]`,… at the same time.

That's ok!

You'll only "override" a value with a better one.

And you'll eventually get to $dist(u, n-1)$

After iteration i, u stores $dist(u, k)$ for some $k \geq i$.

Exit early

If you made it through an entire iteration of the outermost loop and don't update any $dist()$

Then you won't do any more updates in the next iteration either. You can exit early.

More ideas to save constant factors on Wikipedia (or the textbook)

Laundry List of shortest pairs (so far)

Algorithm	Running Time	Special Case	Negative edges?
BFS	$O(m+n)$	ONLY unweighted graphs	X
Simple DP	$O(m+n)$	ONLY for DAGs	X
Dijkstra's	$O(m+n\log n)$		X
Bellman-Ford	$O(mn)$		???

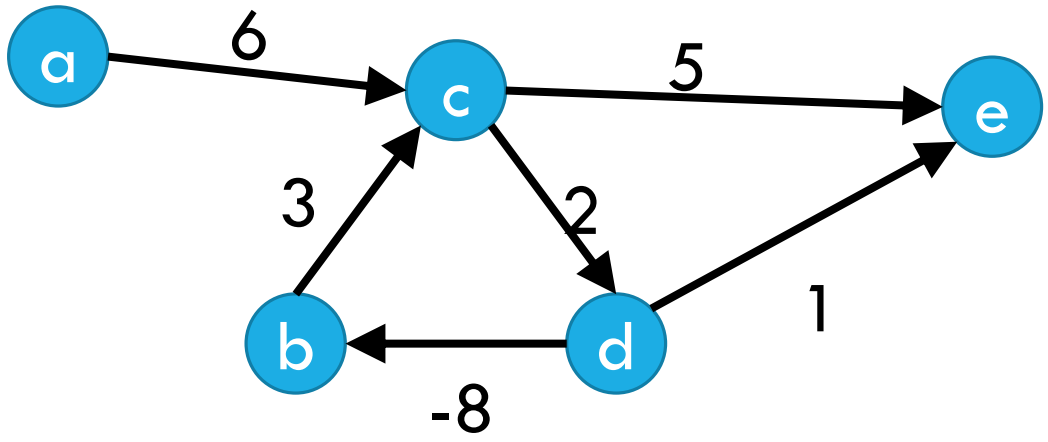
Pseudocode

```
Initialize source.dist=0, u.dist=∞ for others
for(i from 1 to n-1)
   for(every vertex u) //any order
      for(each outgoing edge (u,v))//better!
         if(u.dist+weight(u,v)<v.dist)
            v.dist=u.dist+weight(u,v)
         endIf
      endFor
   endFor
endFor
```

What happens if there's a negative cycle?

Negative Edges

Negative Cycles

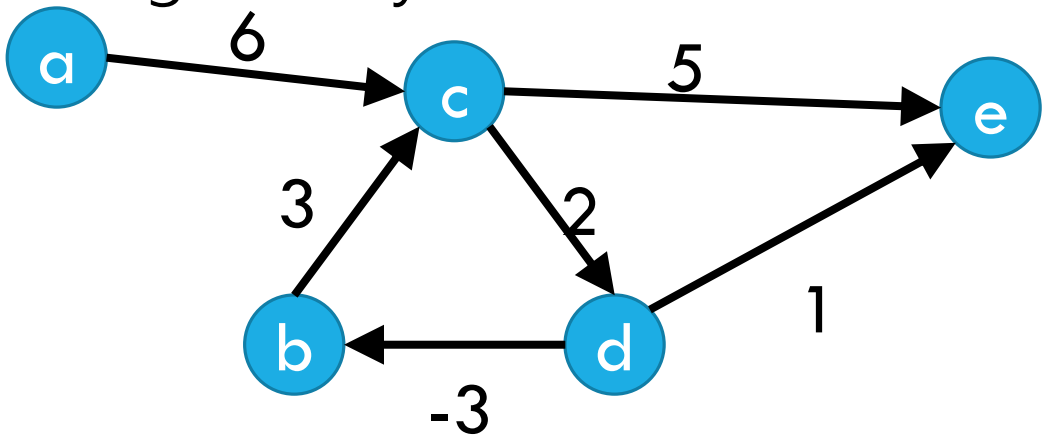


The fastest way from a to e

(i.e. least-weight walk) isn't defined!

No valid answer ($-\infty$)

Negative edges, but only non-negative cycles



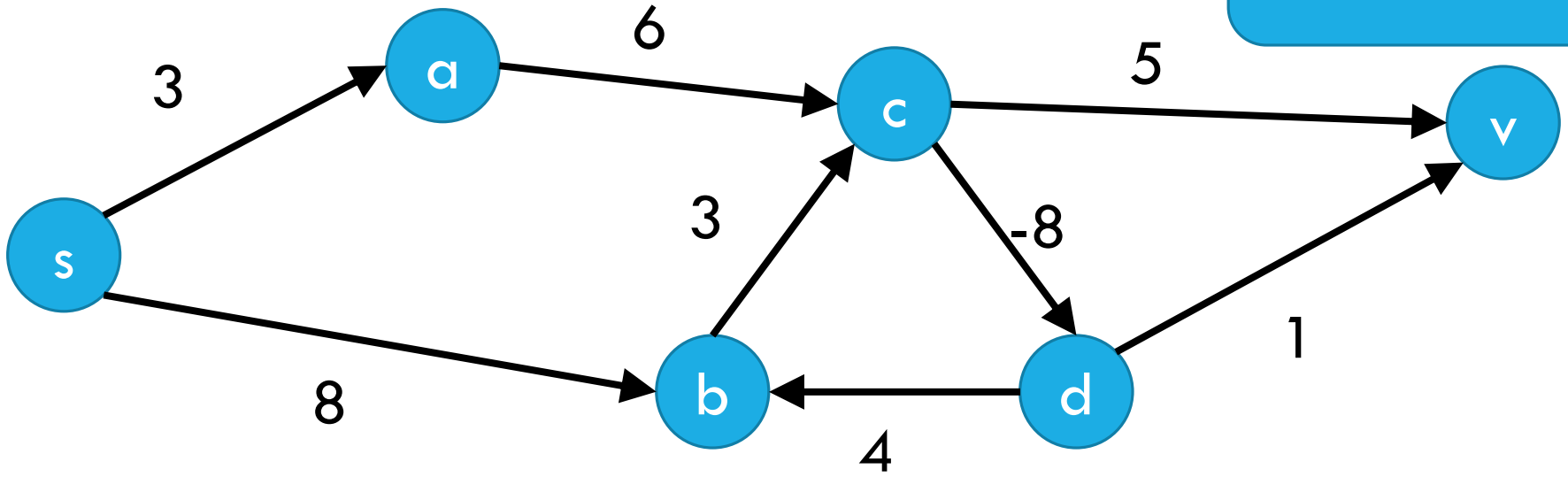
Dijkstra's might fail

But the shortest path IS defined.

There is an answer

Negative Cycle

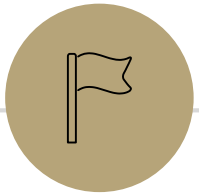
Pollev.com/Robbie



Vertex\i	0	1	2	3	4	5	6
S	0	0	0	0	0		
A	∞	3	3	3	3		
B	∞	8	8	8	5		
C	∞	∞	9	9	9		
D	∞	∞	∞	1	1		
V	∞	∞	∞	14	2		

Laundry List of shortest pairs (so far)

Algorithm	Running Time	Special Case only	Negative edges?
BFS	$O(m+n)$	ONLY unweighted graphs	X
Simple DP	$O(m+n)$	ONLY for DAGs	X
Dijkstra's	$O(m+n\log n)$		X
Bellman-Ford	$O(mn)$		Yes!



All Pairs Shortest Paths

All Pairs

For Dijkstra's or Bellman-Ford we got the distances from the source to every vertex.

What if we want the distances from every vertex to every other vertex?

Another Recurrence

$$dist(v) = \begin{cases} 0 & \text{if } v \text{ is the source} \\ \min_{u:(u,v)\in E} \{dist(u) + weight(u,v)\} & \text{otherwise} \end{cases}$$

Another clever way to order paths.

Put the vertices in some (arbitrary) order $1,2,\ldots,n$

Let $dist(u,v,i)$ be the distance from u to v where the only **intermediate** nodes are $1,2,\ldots,i$

Another Recurrence

Put the vertices in some (arbitrary) order $1,2,\ldots,n$

Let $dist(u,v,i)$ be the distance from u to v where the only **intermediate** nodes are $1,2,\ldots,i$

$$\text{dist}(u,v,i) = \begin{cases} weight(u,v) & \text{if } i = 0, (u,v) \text{ exists} \\ 0 & \text{if } i = 0, u = v \\ \infty & \text{if } i = 0, \text{ no edge } (u,v) \\ \min\{\text{dist}(u,i,i-1) + \text{dist}(i,v,i-1), \text{dist}(u,v,i-1)\} & \text{otherwise} \end{cases}$$

Pseudocode

```
dist[][] = new int[n-1][n-1]
for(int i=0; i<n; i++)
   for(int j=0; j<n; j++)
      dist[i][j] = edge(i,j) ? weight(i,j) : ∞
for(int i=0; i<n; i++)
    dist[i][i] = 0
for every vertex r
   for every vertex u
      for every vertex v
         if(dist[u][r] + dist[r][v] < dist[u][v])
              dist[u][v]=dist[u][r] + dist[r][v]
```

"standard" form of the "Floyd-Warshall" algorithm. Similar to Bellman-Ford, you can get rid of the last entry of the recurrence (only need 2D array, not 3D array).

Running Time

$O(n^3)$

How does that compare to Dijkstra's?

Running Time

If you really want all-pairs…

Could run Dijkstra's n times…

$O(mn \log n + n^2 \log n)$

If $m \approx n^2$ then Floyd-Warshall is faster!

Floyd-Warshall also handles negative weight edges.

Ask Robbie after how to detect them.

Takeaways

Some clever dynamic programming on graphs.

Which library to use (at least asymptotically)?

Need just one source?

Dijkstra's if no negative edge weights.

Bellman-Ford if negative edges.

Need all sources?

Flord-Warshall if negative edges or $m \approx n^2$

Repeated Dijkstra's otherwise

These are all asymptotics! For any "real-world" problem prefer running actual code to see which is faster.