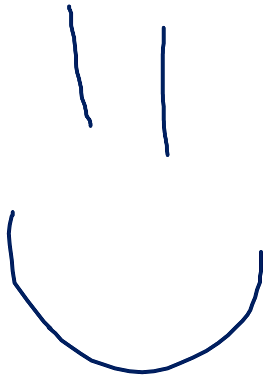


# Still Even More Dynamic Programming

CSE 417 24Wi
Lecture 13

# More Problems

# Maximum Contiguous Subarray Sum

We saw an $O(n \log n)$ divide and conquer algorithm.

Can we do better with DP?

Given: Array $A[]$

Output: $i, j$ such that $A[i] + A[i+1] + \cdots + A[j]$ is maximized.

# Dynamic Programming Process

1. Define the object you're looking for

2. Write a recurrence to say how to find it

3. Design a memoization structure

4. Write an iterative algorithm

# Maximum Contiguous Subarray Sum

We saw an $O(n \log n)$ divide and conquer algorithm.

Can we do better with DP?

Given: Array $A[]$

Output: $i, j$ such that $A[i] + A[i+1] + \cdots + A[j]$ is maximized.

For today: just output the value$A[i] + A[i+1] + \cdots + A[j]$.

Is it enough to know `OPT(i)`?

# Approaching the problem recursively

For DP-style recursion, we're usually looking for an array that's one element smaller.

For today we'll look at arrays with indicies 0,1,2,…$k$.

(where $k < n$)

Define $\underline{OPT(k)}$ to be the sum of the maximum-sum-subarray for the subarray of indicies $\underbrace{0, \dots, k}$. (I.e. answering the problem pretending the array from $k + 1, \dots, n - 1$ doesn't exist).

# Trying to Recurse

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -6 | 3 | 4 | -5 | 2 | 2 | 4 |

7

$OPT(3)$ would give $i$ =2, $j = 3$

$OPT(4)$ would give $i = 2, j = 3$ too

$OPT(7)$ would give $i = 2, j = 7$ – we need to suddenly backfill with a bunch of elements that weren't optimal…

How do we make a decision on index 7? What information do we need?

# What do we need for recursion?

If index $i$ IS going to be included

We need the best subarray **that includes index $\boldsymbol{i-1}$**

If we include anything to the left, we'll definitely include index $i - 1$ (because of the contiguous requirement)

If index $i$ isn't included

We need the best subarray up to $i - 1$, regardless of whether $i - 1$ is included.

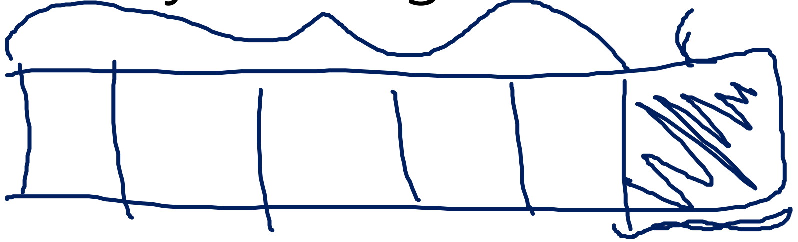
# Two Values

Pollev.com/robbie

Need two recursive values:

$INCLUDE(i)$: sum of the maximum sum subarray among elements from $0$ to $i$ **that includes index $i$** in the sum

$OPT(i)$: sum of the maximum sum subarray among elements $0$ to $i$ (that might or might not include $i$)



How can you calculate these values? Try to write recurrence(s), then think about memoization and running time.

# Recurrences

$$INCLUDE(i) = \begin{cases} \max\{A[i], A[i] + INLCUDE(i-1)\} & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$OPT(i) = \begin{cases} \max\{INCLUDE(i), OPT(i-1)\} & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

If we include $i$, the subarray must be either just $i$ or also include $i-1$.

Overall, we might or might not include $i$. If we don't include $i$, we only have access to elements $i-1$ and before. If we do, we want $INCLUDE(i)$ by definition.

# Example

$A$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -6 | 3 | 4 | -5 | 2 | 2 | 4 |

$OPT(i)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 |  |  |  |  |  |  |  |

$INCLUDE(i)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 |  |  |  |  |  |  |  |

# Example

$A$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -6 | 3 | 4 | -5 | 2 | 2 | 4 |

$OPT(i)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | 5 |  |  |  |  |  |  |

$INCLUDE(i)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -1 |  |  |  |  |  |  |

# Example

$A$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -6 | 3 | 4 | -5 | 2 | 2 | 4 |

$OPT(i)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | 5 | 5 | 7 | 7 | 7 | 7 | 10 |

$INCLUDE(i)$

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -1 | 3 | 7 | 2 | 4 | 6 | 10 |

# Pseudocode

```
int maxSubarraySum(int[] A)
  int n=A.length
  int[] OPT = new int[n]
  int[] Inc = new int[n]
  inc[0]=A[0]; OPT[0] = max{A[0],0}
  for(int i=0;i<n;i++)
    inc[i]=max{A[i], A[i]+inc[i-1]}
    OPT[i]=max{inc[i], opt[i-1]}
  endFor
return OPT[n-1]
```

# Edit Distance

# Edit Distance

More formally:

The edit distance between two strings is:

The minimum number of **deletions, insertions,** and **substitutions** to transform string $x$ into string $y$.

Deletion: removing one character

Insertion: inserting one character (at any point in the string)

Substitution: replacing one character with one other.

# Example

BAB_ x
TAST

What's the distance between `babyyodas` and `tastysoda`?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| B | A | B | | Y | Y | O | D | A | S |
| sub | | sub | ins | | sub | | | | del |
| T | A | S | T | Y | S | O | D | A | |

Distance: 5, one point for each colored box

Quick Checks – can you explain these?

If $x$ has length $n$ and $y$ has length $m$, the edit distance is at most $\max(x, y)$

The distance from $x$ to $y$ is the same as from $y$ to $x$ (i.e. transforming $x$ to $y$ and $y$ to $x$ are the same)

# Finding a recurrence

What information would let us simplify the problem?

What would let us "take one step" toward the solution?

"Handling" one character of $x$ or $y$

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

$OPT(i,j)$ is the edit distance of the strings $x_1x_2\cdots x_i$ and $y_1y_2\cdots y_j$.
(we're indexing strings from 1, it'll make things a little prettier).

# The recurrence

“Handling” one character of $x$ or $y$

i.e. choosing one of insert, delete, or substitution and increasing the “distance” by 1

OR realizing the characters are the same and matching for free.

What does delete look like? $OPT(i-1, j)$ (delete character from $x$ match the rest)

Insert $OPT(i, j-1)$ Substitution: $OPT(i-1, j-1)$

Matching charcters? Also $OPT(i-1, j-1)$ but only if $x_i = y_j$

# The recurrence (v1, we'll improve soon)

"Handling" one character of $x$ or $y$

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

Delete

Insert

Substitution

TODO: Just Match

$$\boxed{OPT(i,j)} = \begin{cases} \min\{\underline{1 + OPT(i-1,j)}, \underline{1 + OPT(i,j-1)}, \underline{1 + OPT(i-1,j-1)}, & \\ j & \text{if } i = 0 \\ i & \text{if } j = 0 \end{cases}$$

# The recurrence (v1, we'll improve soon)

"Handling" one character of $x$ or $y$

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

Delete | Insert | Substitution | Just Match

$$OPT(i,j) = \begin{cases} \min\{\, 1 + OPT(i-1,j), 1 + OPT(i,j-1),\ 1 + OPT(i-1,j-1), OPT(i-1,j-1) + \infty \cdot \mathbb{I}\{x_i \neq y_j\}\} & \\ j & \text{if } i = 0 \\ i & \text{if } j = 0 \end{cases}$$

Idea: only allow "just match" when you can just match.
Otherwise make it ∞ (will never be the min).
In code: if/else branch, probably. This is a math notation trick.

"Indicator" – math for "cast bool to int"

# The recurrence

"Handling" one character of $x$ or $y$

i.e. choosing one of insert, delete, or substitution and increasing the "distance" by 1

OR realizing the characters are the same and matching for free.

$$OPT(i,j) = \begin{cases} \min\{\overset{\boxed{\text{Delete}}}{1 + OPT(i-1,j)}, \overset{\boxed{\text{Insert}}}{1 + OPT(i,j-1)}, \overset{\boxed{\text{Sub and matching}}}{\mathbb{I}[x_i \neq y_j] + OPT(i-1,j-1)}\} & \\ j & \text{if } i = 0 \\ i & \text{if } j = 0 \end{cases}$$

"Indicator" – math for "cast bool to int"

When we could match, we will never substitute; matching will always give us a better score! Still have to check delete, insert (those could be better).

# Recurrence to Code

Just like with Baby Yoda, if you write the recursive code for this "normally" you'll have very slow code. Starting from $OPT(m, n)$, $OPT(m-1, n-1)$ can be reached by:

Delete then insert

Insert then delete

Matching or substitution

**Even worse** than (left, down) or (down, left).

Just like before, memoize the results.

Not a typo! It's memoize, not memorize.

# Edit Distance

| $\mathrm{OPT}(i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | | | |
| T 1 | | | | | | | | | | |
| A 2 | | | | | | | | | | |
| S 3 | | | | | | | | | | |
| T 4 | | | | | | | | | | |
| Y 5 | | | | | | | | | | |
| S 6 | | | | | | | | | | |
| O 7 | | | | | | | | | | |
| D 8 | | | | | | | | | | |
| A 9 | | | | | | | | | | |

# Edit Distance

| | | | |
|---|---|---|---|
| B | A | B | Y |
| T | A | S | |

Current subproblem: edit dist between BABY and TAS

| $\mathrm{OPT}(i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | | | | | | | | | |
| A 2 | 2 | | | | | | | | | |
| S 3 | 3 | | | | | | | | | |
| T 4 | 4 | | | | | | | | | |
| Y 5 | 5 | | | | | | | | | |
| S 6 | 6 | | | | | | | | | |
| O 7 | 7 | | | | | | | | | |
| D 8 | 8 | | | | | | | | | |
| A 9 | 9 | | | | | | | | | |

# Edit Distance

| | | | |
|---|---|---|---|
| B | A | B | Y |
| T | A | S | |

Delete: get rid of Y (cost 1)
BAB
TAS

| OPT($i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | | | | | | | | | |
| A 2 | 2 | | | | | | | | | |
| S 3 | 3 | | | 🟥 | 🟫 | | | | | |
| T 4 | 4 | | | | | | | | | |
| Y 5 | 5 | | | | | | | | | |
| S 6 | 6 | | | | | | | | | |
| O 7 | 7 | | | | | | | | | |
| D 8 | 8 | | | | | | | | | |
| A 9 | 9 | | | | | | | | | |

# Edit Distance

| | | | |
|---|---|---|---|
| B | A | B | Y |
| T | A | S | |

Insert: insert S (cost 1)
BABY
TA

| $\mathrm{OPT}(i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | | | | | | | | | |
| A 2 | 2 | | | | 🟪 | | | | | |
| S 3 | 3 | | | 🟥 | 🟫 | | | | | |
| T 4 | 4 | | | | | | | | | |
| Y 5 | 5 | | | | | | | | | |
| S 6 | 6 | | | | | | | | | |
| O 7 | 7 | | | | | | | | | |
| D 8 | 8 | | | | | | | | | |
| A 9 | 9 | | | | | | | | | |

# Edit Distance

| | | | |
|---|---|---|---|
| B | A | B | Y |
| T | A | S | |

Sub: Transform Y to S (cost 1)
BAB
TA

| $\mathrm{OPT}(i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | | | | | | | | | |
| A 2 | 2 | | | 🟩 | 🟪 | | | | | |
| S 3 | 3 | | | 🟥 | 🟫 | | | | | |
| T 4 | 4 | | | | | | | | | |
| Y 5 | 5 | | | | | | | | | |
| S 6 | 6 | | | | | | | | | |
| O 7 | 7 | | | | | | | | | |
| D 8 | 8 | | | | | | | | | |
| A 9 | 9 | | | | | | | | | |

# Edit Distance

| | | | |
|---|---|---|---|
| B | A | B | Y |
| T | A | S | |

Gold entry will be min of:
$1 +$ delete
$1 +$ insert
$1 +$ sub

| $\mathrm{OPT}(i, j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | | | | | | | | | |
| A 2 | 2 | | | 🟩 | 🟪 | | | | | |
| S 3 | 3 | | | 🟥 | 🟨 | | | | | |
| T 4 | 4 | | | | | | | | | |
| Y 5 | 5 | | | | | | | | | |
| S 6 | 6 | | | | | | | | | |
| O 7 | 7 | | | | | | | | | |
| D 8 | 8 | | | | | | | | | |
| A 9 | 9 | | | | | | | | | |

# Edit Distance

$1 + 2$
$1 + 3$
$1 + 2$
Min: 3

Gold entry will be min of:
$1 +$ delete
$1 +$ insert
$1 +$ sub

| $\mathbf{OPT}(i, j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| A 2 | 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| S 3 | 3 | 3 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| T 4 | 4 | 4 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| Y 5 | 5 | 5 | 4 | 4 | 3 |  |  |  |  |  |
| S 6 |  |  |  |  |  |  |  |  |  |  |
| O 7 |  |  |  |  |  |  |  |  |  |  |
| D 8 |  |  |  |  |  |  |  |  |  |  |
| A 9 |  |  |  |  |  |  |  |  |  |  |

# Edit Distance

Fill in the next two entries. Be careful with the sub/match distinction!

| $\mathrm{OPT}(i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| A 2 | 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| S 3 | 3 | 3 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| T 4 | 4 | 4 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| Y 5 | 5 | 5 | 4 | 4 | 3 |  |  |  |  |  |
| S 6 |  |  |  |  |  |  |  |  |  |  |
| O 7 |  |  |  |  |  |  |  |  |  |  |
| D 8 |  |  |  |  |  |  |  |  |  |  |
| A 9 |  |  |  |  |  |  |  |  |  |  |

# Edit Distance

Fill in the next two entries. Be careful with the sub/match distinction!

| OPT$(i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| A 2 | 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| S 3 | 3 | 3 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| T 4 | 4 | 4 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| Y 5 | 5 | 5 | 4 | 4 | 3 | **3** | **4** |  |  |  |
| S 6 |  |  |  |  |  |  |  |  |  |  |
| O 7 |  |  |  |  |  |  |  |  |  |  |
| D 8 |  |  |  |  |  |  |  |  |  |  |
| A 9 |  |  |  |  |  |  |  |  |  |  |

Y's match, so sub is free!

# Edit Distance

| $\mathrm{OPT}(i, j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| A 2 | 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| S 3 | 3 | 3 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| T 4 | 4 | 4 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| Y 5 | 5 | 5 | 4 | 4 | 3 | 3 | 4 | 5 | 6 | 7 |
| S 6 | 6 | 6 | 5 | 5 | 4 | 4 | 4 | 5 | 6 | 6 |
| O 7 | 7 | 7 | 6 | 6 | 5 | 5 | 4 | 5 | 6 | 7 |
| D 8 | 8 | 8 | 7 | 7 | 6 | 6 | 5 | 4 | 5 | 6 |
| A 9 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 6 | 4 | 5 |

# What if we want the list if inserts,delete,subs?

Or with Baby Yoda the actual path he has to go?

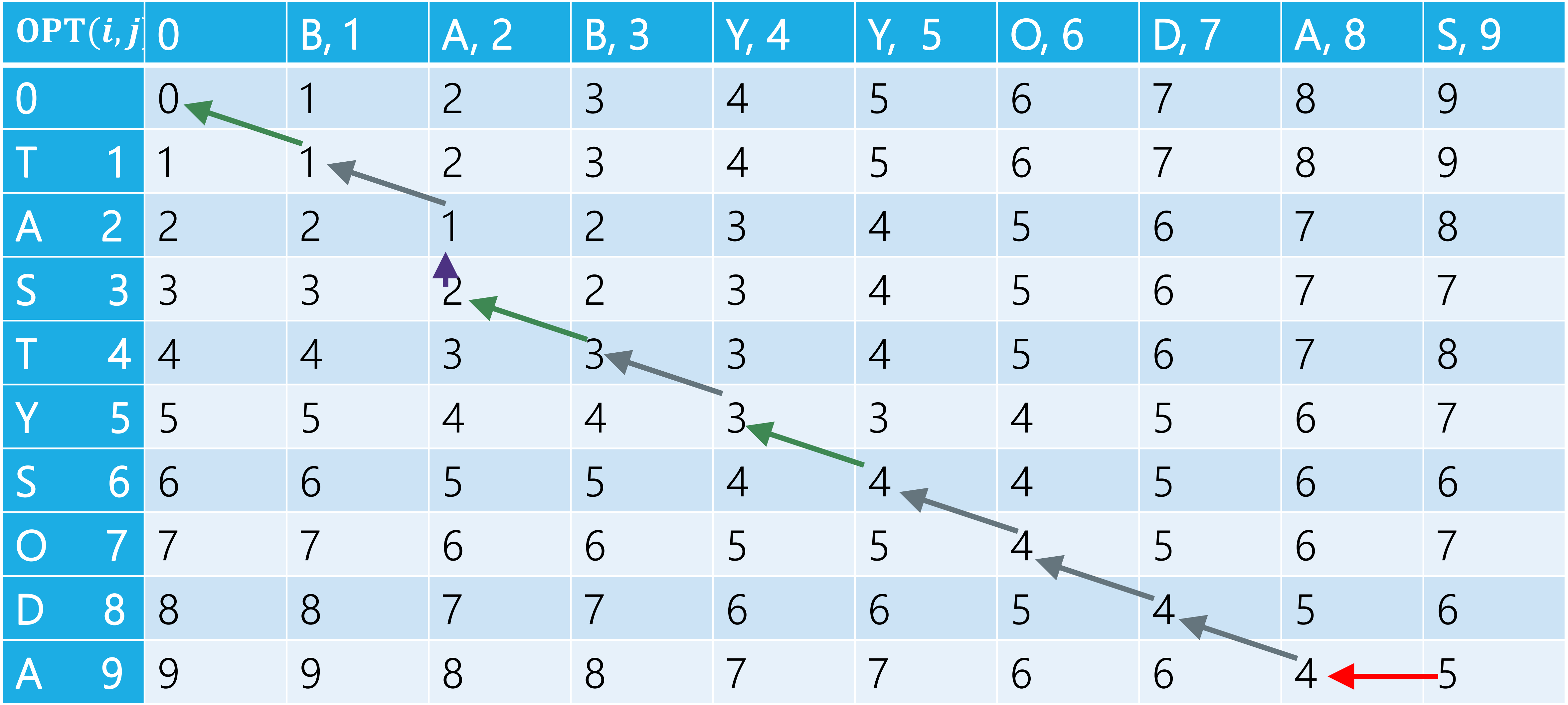
You can always find it. Just ask “well which recursive call was the one I used?” (which one was the minimum, in this problem)

If $OPT(i-1, j)$ was the minimum, then that means you should delete!

You can pretty much always find “the object” this way.

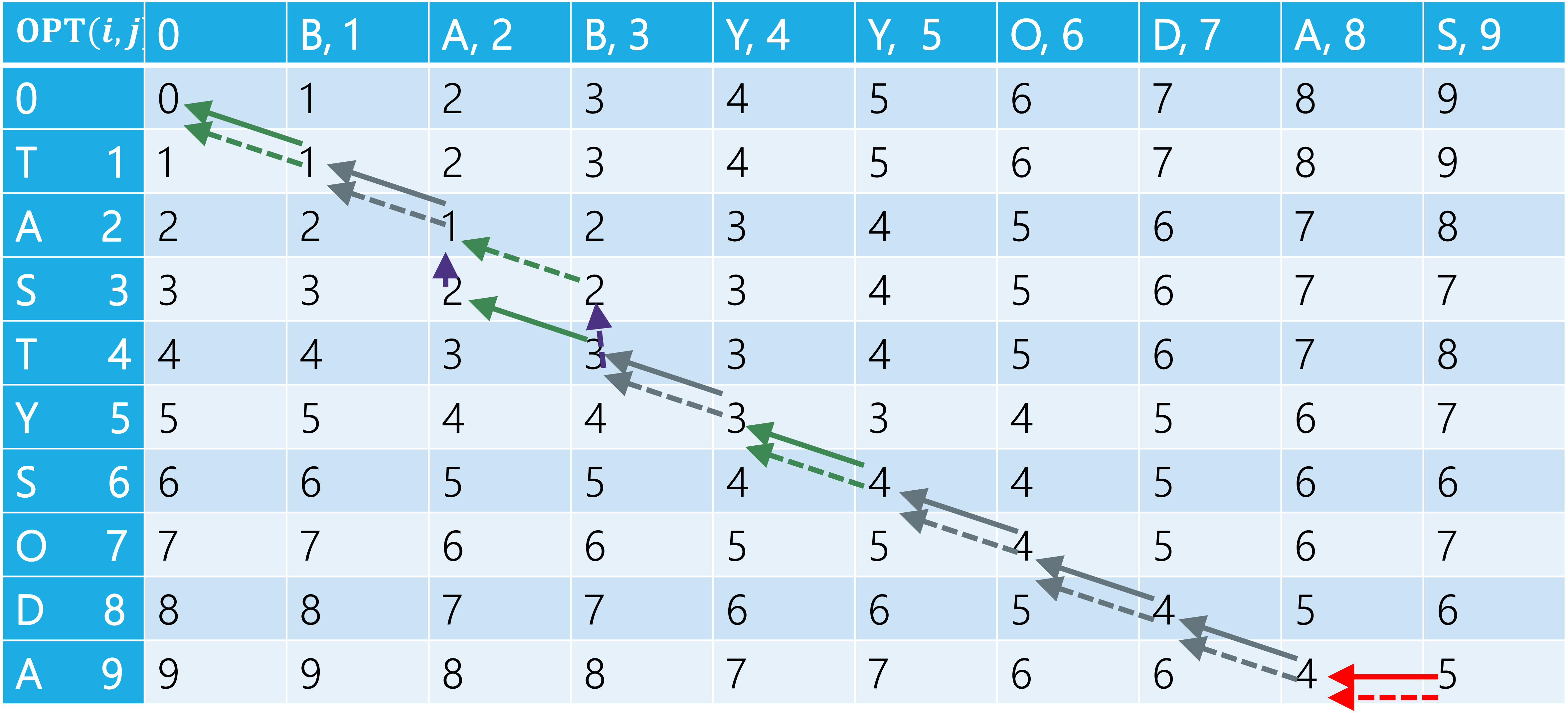
On a future homework, a problem asks you to write the bookkeeping code. For lecture/most problems we’re going to just find the number.

# Edit Distance



| $\mathrm{OPT}(i,j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| A 2 | 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| S 3 | 3 | 3 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| T 4 | 4 | 4 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| Y 5 | 5 | 5 | 4 | 4 | 3 | 3 | 4 | 5 | 6 | 7 |
| S 6 | 6 | 6 | 5 | 5 | 4 | 4 | 4 | 5 | 6 | 6 |
| O 7 | 7 | 7 | 6 | 6 | 5 | 5 | 4 | 5 | 6 | 7 |
| D 8 | 8 | 8 | 7 | 7 | 6 | 6 | 5 | 4 | 5 | 6 |
| A 9 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 6 | 4 | 5 |

# Edit Distance



| OPT($i, j$ | 0 | B, 1 | A, 2 | B, 3 | Y, 4 | Y, 5 | O, 6 | D, 7 | A, 8 | S, 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| T 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| A 2 | 2 | 2 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| S 3 | 3 | 3 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| T 4 | 4 | 4 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| Y 5 | 5 | 5 | 4 | 4 | 3 | 3 | 4 | 5 | 6 | 7 |
| S 6 | 6 | 6 | 5 | 5 | 4 | 4 | 4 | 5 | 6 | 6 |
| O 7 | 7 | 7 | 6 | 6 | 5 | 5 | 4 | 5 | 6 | 7 |
| D 8 | 8 | 8 | 7 | 7 | 6 | 6 | 5 | 4 | 5 | 6 |
| A 9 | 9 | 9 | 8 | 8 | 7 | 7 | 6 | 6 | 4 | 5 |

# Dynamic Programming Process

1. Define the object you're looking for

   OPT(i,j) is the minimum number of insertions, deletions, and substitutions required to transform $x_1 \dots x_i$ to $y_1 \dots y_j$

2. Write a recurrence to say how to find it

   ☑

3. Design a memoization structure

   $m \times n$ Array

4. Write an iterative algorithm

   Outer loop: increasing $i$ (i.e., row-by-row starting from 1)
   Inner loop: increasing $j$ (i.s., column-by-column starting from 1)

# Some General Advice

# Recursive Thinking In General

As before, the hardest part is designing the recurrence.

It sometimes helps to think from multiple different angles.

**Top-down:** What's the first step to take?

Baby Yoda will first go left or down. Use recursion to find out which of left or down is better.

The farthest right operation in the string transformation will be one of insert, delete, substitute, match for free. Use recursion to find out which is best.

# Recursive Thinking In General

**Bottom-Up:** What information could a recursive call give me that would help?

How does a path through most of the map help Baby Yoda?

Well we just need to know the values one left and one down.

The edit distance between which strings would help us compute the edit distance between our strings?

Well if we know the distance between $x_1 \dots x_{i-1}$ and $y_1 \dots y_{j-1}$ then that would tell us what happens if we substitute…that might lead you to insertions and deletions too.

# Recursive Thinking In General

Some people refer to the “Optimal Substructure Property”

From the optimum (most eggs, fewest number of string operations) for a slightly smaller problem (Baby Yoda starting closer to the end, slightly smaller strings), we need to be able to build up the optimum for the full problem.

# Longest Increasing Subsequence

# Longest Increasing Subsequence

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -6 | 3 | 6 | -5 | 2 | 8 | 10 |

Longest set of (not necessarily consecutive) elements that are increasing

5 is optimal for the array above

(indices $1,2,3,6,7$; elements $-6,3,6,8,10$)

For simplicity – assume all array elements are distinct.

# Longest Increasing Subsequence

What do we need to know to decide on element $i$?

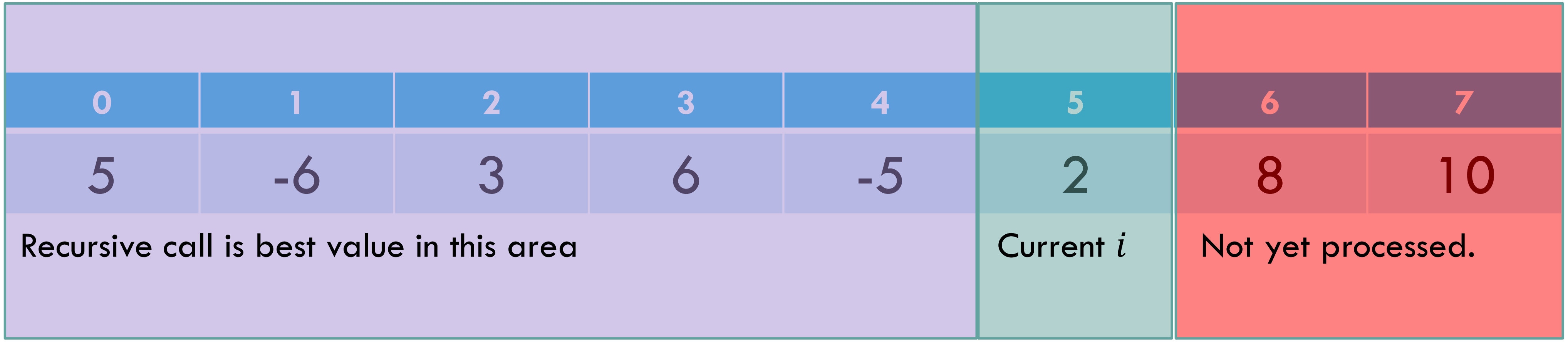
Is it allowed?

Will the sequence still be increasing if it's included?

Still thinking right to left --

Two indices: index we're looking at, and index of upper bound on elements (i.e. the value we need to decide if we're still increasing).

# Recurrence



| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 5 | -6 | 3 | 6 | -5 | 2 | 8 | 10 |

Recursive call is best value in this area

Current $i$

Not yet processed.

Need recursive answer to the left

Currently processing $i$

Recursive calls to the left are needed to know optimum from $1 \ldots i$

Will move $i$ to the right in our iterative algorithm

# Longest Increasing Subsequence

$LIS(i, j)$ is “Number of elements of the maximum increasing subsequence from $0, \ldots, i$ where every element of the sequence is at most $A[j]$”

Need a recurrence

$$LIS(i, j) = \begin{cases} ? & \text{if } i < 0 \\ ? & \text{if } i = 0 \\ ? & \text{if } A[i] > A[j] \\ ? & \text{otherwise} \end{cases}$$

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If $A[i] > A[j]$ element $i$ cannot be included in an increasing subsequence where every element is at most $A[j]$. So taking the largest among the first $i - 1$ suffices.

If $A[i] \leq A[j]$, then if we include $i$, we may include elements to the left only if they are less than $A[i]$ (since $A[i]$ will now be the last, and therefore largest, of elements $1 \ldots i$. If we don't include $i$ we want the maximum increasing subsequence among $1 \ldots i - 1$.

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$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

If $A[i] > A[j]$ element $i$ cannot be included in an increasing subsequence where every element is at most $A[j]$. So taking the largest among the first $i-1$ suffices.

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Memoization structure? $n \times n$ array.

Filling order?

# LIS

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

$j$

$i$

| | 0, 5 | 1, −6 | 2, 3 | 3, 6 | 4, −5 | 5, 2 | 6, 8 | 7, 10 |
|---|---|---|---|---|---|---|---|---|
| 0, 5 | | | | | | | | |
| 1, −6 | | | | | | | | |
| 2, 3 | | | | | | | | |
| 3, 6 | | | 🟫 | | 🟫 | | | |
| 4, −5 | | | 🟪 | | | | | |
| 5, 2 | | | | | | | | |
| 6, 8 | | | | | | | | |
| 7, 10 | | | | | | | | |

# LIS

$LIS(1,0)$ $A[1] < A[0]$ not allowed:
Take $LIS(0,0)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \le A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

$j$

$i$

| | 0, 5 | 1, −6 | 2, 3 | 3, 6 | 4, −5 | 5, 2 | 6, 8 | 7, 10 |
|---|---|---|---|---|---|---|---|---|
| 0, 5 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1, −6 | 1 | | | | | | | |
| 2, 3 | | | | | | | | |
| 3, 6 | | | | | | | | |
| 4, −5 | | | | | | | | |
| 5, 2 | | | | | | | | |
| 6, 8 | | | | | | | | |
| 7, 10 | | | | | | | | |

# LIS

$LIS(1,1)$ $A[1] \leq A[1]$ can add, $1 + LIS(0,1)$ or $LIS(0,1)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

$j$

$i$

| | 0, $5$ | 1, $-6$ | 2, $3$ | 3, $6$ | 4, $-5$ | 5, $2$ | 6, $8$ | 7, $10$ |
|---|---|---|---|---|---|---|---|---|
| 0, $5$ | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1, $-6$ | 1 | 1 | | | | | | |
| 2, $3$ | | | | | | | | |
| 3, $6$ | | | | | | | | |
| 4, $-5$ | | | | | | | | |
| 5, $2$ | | | | | | | | |
| 6, $8$ | | | | | | | | |
| 7, $10$ | | | | | | | | |

# LIS

$LIS(1,2)$ $A[1] \leq A[2]$ allowed to add:
$1 + LIS(0,1)$ or $LIS(0,2)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

$j$

$i$

| | 0, 5 | 1, −6 | 2, 3 | 3, 6 | 4, −5 | 5, 2 | 6, 8 | 7, 10 |
|---|---|---|---|---|---|---|---|---|
| 0, 5 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1, −6 | 1 | 1 | 1 | | | | | |
| 2, 3 | | | | | | | | |
| 3, 6 | | | | | | | | |
| 4, −5 | | | | | | | | |
| 5, 2 | | | | | | | | |
| 6, 8 | | | | | | | | |
| 7, 10 | | | | | | | | |

# LIS

$LIS(2{,}0)$ $A[2] \leq A[0]$ allowed to add
$1 + LIS(1{,}2)$ or $LIS(1{,}0)$

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

$j$

$i$

| | 0, 5 | 1, −6 | 2, 3 | 3, 6 | 4, −5 | 5, 2 | 6, 8 | 7, 10 |
|---|---|---|---|---|---|---|---|---|
| 0, 5 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1, −6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2, 3 | 2 | | | | | | | |
| 3, 6 | | | | | | | | |
| 4, −5 | | | | | | | | |
| 5, 2 | | | | | | | | |
| 6, 8 | | | | | | | | |
| 7, 10 | | | | | | | | |

# LIS

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 0 \\ \mathbb{I}[A[i] \leq A[j]] & \text{if } i = 0 \\ LIS(i-1,j) & \text{if } A[i] > A[j] \\ \max\{1 + LIS(i-1,i), LIS(i-1,j)\} & \text{otherwise} \end{cases}$$

$j$

$i$

| | 0, 5 | 1, $-6$ | 2, 3 | 3, 6 | 4, $-5$ | 5, 2 | 6, 8 | 7, 10 |
|---|---|---|---|---|---|---|---|---|
| 0, 5 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| 1, $-6$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2, 3 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 2 |
| 3, 6 | 2 | 1 | 2 | 3 | 1 | 1 | 3 | 3 |
| 4, $-5$ | 2 | 1 | 2 | 3 | 2 | 2 | 3 | 3 |
| 5, 2 | 3 | 1 | 3 | 3 | 2 | 3 | 3 | 3 |
| 6, 8 | 3 | 1 | 3 | 3 | 2 | 3 | 4 | 4 |
| 7, 10 | 3 | 1 | 3 | 3 | 2 | 3 | 4 | 5 |

# pseudocode

```
//real code snippet that actually generated the table on the last slide
for(int j=0; j < n; j++){
     vals[0][j] = (A[0] <= A[j]) ? 1 : 0;
}
for(int i = 1; i < 8; i++){
     for(int j = 0; j < n; j++){
           if(A[i] > A[j])
                 vals[i][j] = vals[i-1][j];
           else{
                 vals[i][j] = Math.max(1+vals[i-1][i], vals[i-1][j]);
           }
     }
}
```