Linear Programming Sample Problem

1. Production Planning

You are the manager of a company that sells handwoven carpets, a product for which the demand is seasonal. Our analyst has obtained demand estimates for all the months of the next year $d_1, d_2, \ldots, d_{12}$. As expected, said demands are incredibly uneven within the range of 450 to 940.

For some actual information about the company: You oversee 30 employees, each of whom makes about 20 carpets a month in exchange for a monthly salary of $\$2000$. There is no initial surplus of carpets.

However, this situation is not disastrous; you know that the company is equipped to deal with fluctuations in demand in three key ways.

- **Overtime:** This is expensive since you have to pay employees about 70% more than their regular pay, however, you are guaranteed that workers aren't eligible for overtime pay beyond 30%.

- **Hiring and firing workers:** This costs $\$300$ and $\$400$ dollars respectively.

- **Storing surplus production:** This costs $\$8$ a month.

An important thing to note is that the company starts the year with no surplus stored markets and would like to end the year in the same way (if possible).

(a) What are your variables?

Solution:

> We define our variables as
>
> - $w_i :=$ number of workers during the ith month, this starts off as being 30 ($w_0 = 30$).
> - $x_i :=$ number of carpets made during the ith month.
> - $o_i :=$ number of carpets made during overtime in month i.
> - $h_i, f_i :=$ number of workers hired and fired, respectively at the beginning of month i.
> - $s_i :=$ number of carpets stored at the end of month i ($s_0 = 0$).

(b) What are your constraints?

Solution:

> Firstly, all of our variables need to be non-negative
>
> $$w_i, x_i, o_i, f_i, s_i \geq 0 \text{ for } i \in \{1, 2, \ldots, 12\}$$
>
> The total number of carpets made in a month is simply the number of carpets made during regular production and the additional carpets made during overtime
>
> $$x_i = 20w_i + o_i \text{ for } i \in \{1, 2, \ldots, 12\}$$
>
> Also note that the number of workers in a given month can change (based on who gets hired or fired)
>
> $$w_i = w_{i-1} + h_i - f_i$$
>
> The number of carpets stored at the end of a month is simply the number of carpets we started with plus

the number of carpets we made minus demand for the month

$$s_i = s_{i-1} + x_i - d_i$$

Finally note that the total number of carpets made during overtime is limited. We have that o_i represents the number of carpets made by workers in overtime sessions and as per one of the constraints in the problem, overtime is capped at 30%, therefore

$$o_i \leq \frac{30}{100} \text{ (number of carpets made normally)} = \frac{30}{100} \cdot \left(\underbrace{20}_{\text{number of carpets per worker}} \cdot w_i \right) = 6w_i$$

so

$$o_i \leq 6w_i$$

(c) What is your objective function?

Solution:

Our objective function is of the form

$$\min \left(2000 \sum_i w_i + 300 \sum_i h_i + 400 \sum_i f_i + 8 \sum_i s_i + 170 \sum_i o_i \right)$$

Note that the last term comes from the fact that

$$\text{earnings from overtime} = \underbrace{\frac{2000}{20}}_{\text{profit per carpet}} \times \underbrace{(1 + 0.7)}_{\text{total pay and overtime pay}} \times \sum_i o_i = 170 \sum_i o_i$$