

# diction vs positive

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## 1 Introduction

It's often easy to get confused between the two proof techniques, namely that of proof by contradiction and proof by contrapositive; in fact, they have the same first prefix! (Hopefully, you appreciate the humor in the title of this document as much as I did). The following excerpt hopefully can help clarify some of the differences and even similarities to understand how exactly the machinery involved is **different** in both cases.

To define and properly distinguish between the two notions, it may help to see some explicit definitions of the terms involved.

**Definition 1.** An **implication** is a statement of the form  $p \rightarrow q$ , where  $p$  and  $q$  are statements. The statement  $p$  is called the **hypothesis** and the statement  $q$  is called the **conclusion**.

1. For example, the statement “If  $x$  is a real number and  $x^2 = 4$ , then  $x = 2$  or  $x = -2$ ” is an implication.
2. Another example of an implication (which is false) is “If  $x$  is a real number then  $x^2 = 4$ ”.

**Fact 1.** It helps to realize that another way of writing an implication  $p \rightarrow q$  is by writing it down in the form  $\neg p \vee q$ . Note that it's not important to understand why these two are equivalent, or where that equivalence comes from, but rather to understand some of the symbols involved.  $\neg$  is the symbol for **negation** and  $\vee$  is the symbol for **or**, (also called disjunction).

1. For example, the negation of the statement  $x > 0$  is  $x \leq 0$  and the disjunction of the statements  $x > 0$  and  $x \leq 0$  is  $x > 0 \vee x \leq 0$  which is always true!
2. For an implication of the form  $p \rightarrow q$ , we know that it can be equivalently written as  $\neg p \vee q$ . The negation of this claim is  $\neg(\neg p \vee q)$  which is equivalent to  $p \wedge \neg q$  where the symbol  $\wedge$  is the symbol for **and**, (i.e. taking the **and** of two statements). (again it's not important that you understand the simplification going on here but rather how this can potentially help you).
3. If I were given the implication “If  $x$  is even, then  $x^2$  is even”, then the negation of this statement is “ $x$  is even and  $x^2$  is odd”.

## 2 Proof by Contradiction

Sometimes it is easier to prove a statement by assuming the **negation** of said statement and then showing that this assumption leads to a *contradiction*. Why is this useful? Well, the idea is that it should be impossible to prove false statements, so the hope is that if we can achieve such a thing, it must have been that we erred in the beginning of our journey, i.e. in the assumption we made.

### 2.1 How to structure a proof by contradiction

The following list has been adapted from a document by Stanford linked [here](#).

1. This may seem like the least important but for the sake of clarity it always helps to declare that you're using a proof by contradiction. This can be done by saying something like "We will prove the statement by contradiction" or "We will prove the statement by assuming the negation and showing that this leads to a contradiction". There's a well known acronym you may want to use here: AFSOC (Assume for the sake of contradiction). It is always better to be more explicit than less in your writing. It does not hurt to actually write out the negation of the statement you're trying to prove. This helps us while reading the proof to understand what kind of assumptions you're making.
2. Start with the assumption you made above and following a logical sequence of steps, show that something impossible or known to be false occurs.
3. State the contradiction you've reached and if it's not obvious, explain why it's a contradiction.
4. Conclude the proof by saying that the assumption you made must have been false and hence the original statement must be true.

Statement you want to prove	Negation of the statement
All P's are Q's	There exists a P that is not a Q
No P's are Q's	There exists a P that is a Q
Some P's are Q's	All P's are not Q's
Some P's are not Q's	All P's are Q's
If P is true then Q is true	P is true and Q is not true
P is true and Q is true	P is false, or Q is false, or both are false
P is true or Q is true, or both are true	P is false and Q is false

**Theorem 1.** If  $a, b$  are integers, then  $a^2 - 4b \neq 2$ .

**Proof:**

Suppose for the sake of contradiction that  $a, b$  are integers such that  $a^2 - 4b = 2$ . Then we have that  $a^2 = 4b + 2 = 2(2b + 1)$ . Since  $2b + 1$  is an integer, we have that  $a^2$  is even. By Theorem 4, this means that  $a$  is even and thereby there exists some integer  $m$  such that  $a = 2m$ . Then we have that

$$a = 2m \implies a^2 - 4b = 2 \implies (2m)^2 - 4b = 2 \implies 4m^2 - 4b = 2 \implies 2m^2 - 2b = 1. \implies 2(m^2 - b) = 1.$$

However this must mean that 1 is even, which is a contradiction. This completes the proof. ■

**Theorem 2.** For all integers  $n$ , i.e.  $n \in \mathbb{Z}$ , if  $n^3 + 5$  is odd, then  $n$  is even.

**Proof:**

We present below a proof by contradiction. Let  $n$  be an integer such that  $n^3 + 5$  is odd. Suppose for the sake of contradiction that  $n$  is odd. Then  $n = 2k + 1$  for some integer  $k$ . However, since  $n^3 + 5$  is odd, there exists some integer  $m$  such that  $n^3 + 5 = 2m + 1$ . Then we have that

$$2m + 1 = n^3 + 5$$

$$2m + 1 = (2k + 1)^3 + 5$$

$$= 8k^3 + 12k^2 + 6k + 1 + 5$$

$$2m = 8k^3 + 12k^2 + 6k + 5 \quad \text{[subtracting 1 from both sides]}$$

$$m = 4k^3 + 6k^2 + 3k + 5/2 \quad \text{[dividing both sides by 2]}$$

$$m - 4k^3 - 6k^2 - 3k = 5/2 \quad \text{[subtracting } 4k^3 + 6k^2 + 3k \text{ from both sides].}$$

However this is impossible since the left hand side is a difference of products of integers, which we know to be integers, and the right hand side is not an integer. Hence we have reached a contradiction. Therefore, our assumption that  $n$  is odd must have been false and hence  $n$  is even. This completes the proof. ■

**Theorem 3.**  $\sqrt{2}$  is irrational.

**Proof:**

Suppose once again for the sake of contradiction that  $\sqrt{2}$  is rational. Then by the definition of rationality there exist integers  $a$  and  $b$  such that  $\sqrt{2} = a/b$ . Without loss of generality, we can assume that this fraction is in simplest form, i.e. that  $a$  and  $b$  have no common factors (since we can always divide out any common factors). Then we have that

$$\sqrt{2} = \frac{a}{b} \implies 2 = \frac{a^2}{b^2} \implies 2b^2 = a^2.$$

This implies that by definition  $a^2$  is even, and hence by Theorem 4  $a$  is even. Therefore there exists some integer  $m \in \mathbb{Z}$  such that  $a = 2m$ . Then

$$2b^2 = a^2 = (2m)^2 = 4m^2 \implies b^2 = 2m^2.$$

This means that by definition  $b^2$  is even, and hence by the same theorem as before,  $b$  is even. However, this is a contradiction since we assumed that  $a$  and  $b$  had no common factors, but now we have that they are both even and therefore have a common factor of 2. This must mean that our assumption that  $\sqrt{2}$  is rational is false. This completes the proof. ■

### 3 Proof by Contrapositive

Recall that any statement or implication has a contrapositive. The contrapositive of an implication  $p \rightarrow q$  is  $\neg q \rightarrow \neg p$ . In words this is just saying that if  $q$  is false then  $p$  is false. Sometimes when faced with statements of this form, and you're having some difficulty proving the statement directly, it may be easier to prove the contrapositive.

#### 3.1 How to structure a proof by contrapositive

The following list has been adapted from a document by Stanford linked [here](#).

1. Once again start by declaring that you're going to prove the contrapositive of the statement you wish to prove. This can be done by saying something like "We will prove the contrapositive of the statement" or "By contrapositive, we will instead show this equivalent statement". Again it's super important to not overlook this step since it helps the reader (us!) understand what you're trying to do, and in writing the details out, it reduces the chances of you taking the wrong path.
2. You can now use **any** proof technique you'd like to prove the contrapositive of the statement. Often times this will be a proof that proceeds as expected, i.e. you will start with the assumption that the conclusion of the contrapositive is false and then show that the hypothesis of the contrapositive is also false.

**Theorem 4.** Let  $n$  be an integer. If  $n^2$  is even, then  $n$  is even.

**Proof:**

We will prove the contrapositive of the statement. That is, we will show that if  $n$  is odd, then  $n^2$  is odd. Suppose that  $n$  is odd. Then there exists some integer  $m$  such that  $n = 2m + 1$ . Then we have that

$$n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1.$$

Since  $2m^2 + 2m$  is an integer, we have that  $n^2$  is odd. This completes the proof. ■

**Theorem 5.** For all integers  $m, n$  if  $mn$  is even, then  $m$  is even or  $n$  is even.

**Proof:**

We will prove the contrapositive of the statement. That is, we will show that if  $m$  is odd and  $n$  is odd, then  $mn$  is odd. Suppose that  $m$  is odd and  $n$  is odd. Then there exists some integers  $a, b$  such that  $m = 2a + 1$  and  $n = 2b + 1$ . Then we have that

$$mn = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1.$$

Since  $2ab + a + b$  is an integer, we have that  $mn$  is odd. This completes the proof. ■

**Theorem 6.** Let  $x, y$  be two integers. Suppose  $x^2(y^2 - 2y)$  is odd. Prove that  $x$  and  $y$  are odd

**Proof:**

We will prove the contrapositive of the statement. That is, we will show that if  $x$  is even or  $y$  is even, then  $x^2(y^2 - 2y)$  is even. There are two possible cases to consider.

1. Suppose  $x$  is even. Then there exists some integer  $m$  such that  $x = 2m$ . Then we have that

$$x^2(y^2 - 2y) = (2m)^2(y^2 - 2y) = 4m^2(y^2 - 2y) = 2(2m^2(y^2 - 2y)).$$

Since  $2m^2(y^2 - 2y)$  is an integer, we have that  $x^2(y^2 - 2y)$  is even.

2. Suppose  $y$  is even. Then there exists some integer  $n$  such that  $y = 2n$ . Then we have that

$$x^2(y^2 - 2y) = x^2(4n^2 - 4n) = 4x^2n^2 - 4x^2n = 4(x^2n^2 - x^2n).$$

Since  $x^2n^2 - x^2n$  is an integer, we have that  $x^2(y^2 - 2y)$  is even.

In both cases we have shown that if  $x$  is even or  $y$  is even, then  $x^2(y^2 - 2y)$  is even. This completes the proof. ■

## 4 References

- [A guide to indirect proofs: Lee, Schwarz](#)
- [Proof techniques: 311](#)