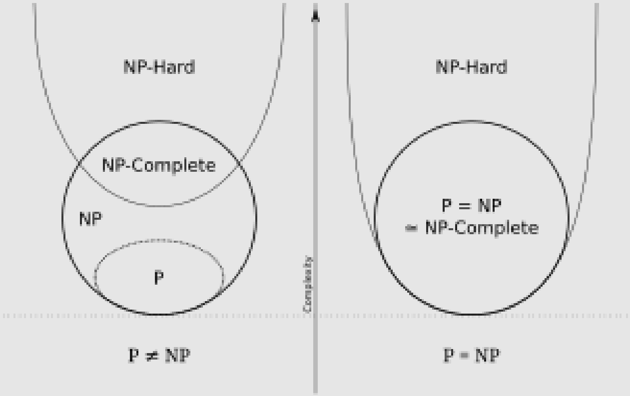
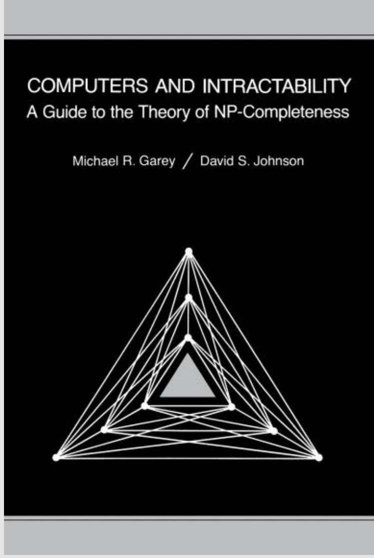




CSE 417
Algorithms and Complexity

Autumn 2024

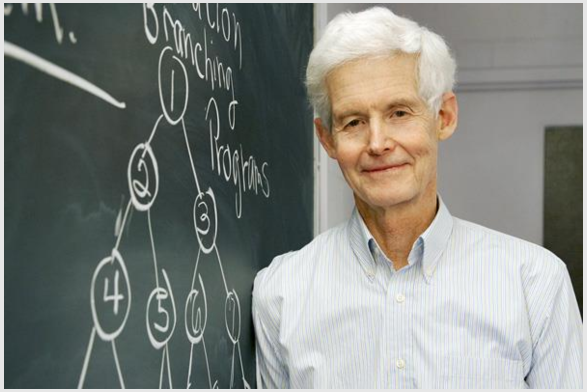
Lecture 27

NP-Completeness I

Announcements

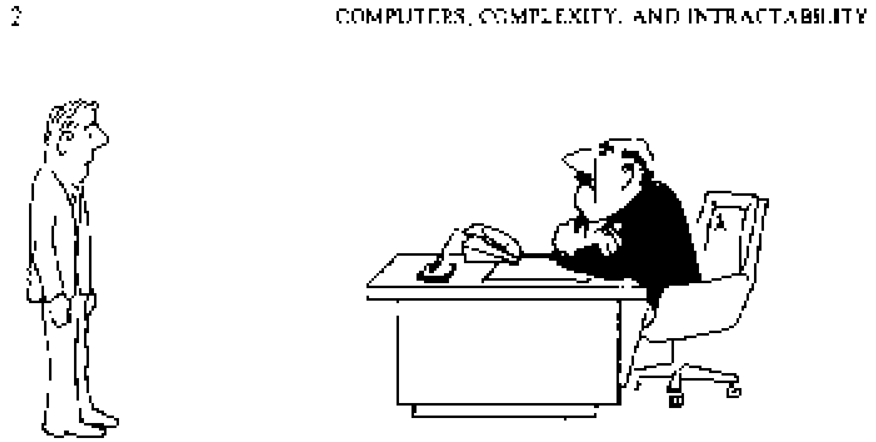
- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, December 9, 8:30 AM
  - One Hour Fifty Minutes
  - Closed book, no notes

Mon, Dec 2	NP-Completeness
Wed, Dec 4	NP-Completeness
Fri, Dec 6	Last Lecture: NP-Completeness and Beyond
Mon, Dec 9	Final Exam

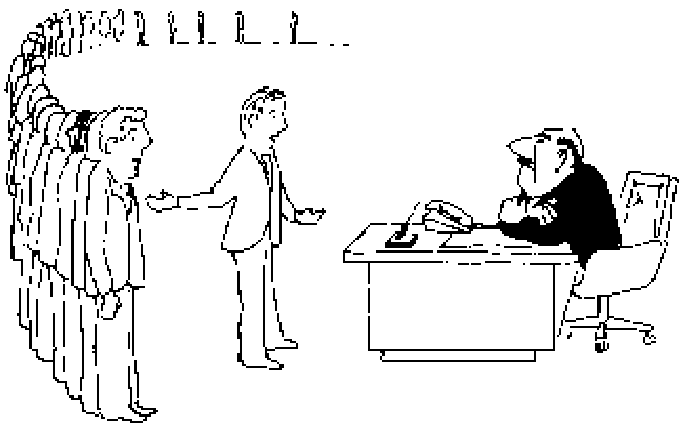


NP-Completeness

NP Completeness



I can't find an efficient algorithm, I guess I'm just too dumb.



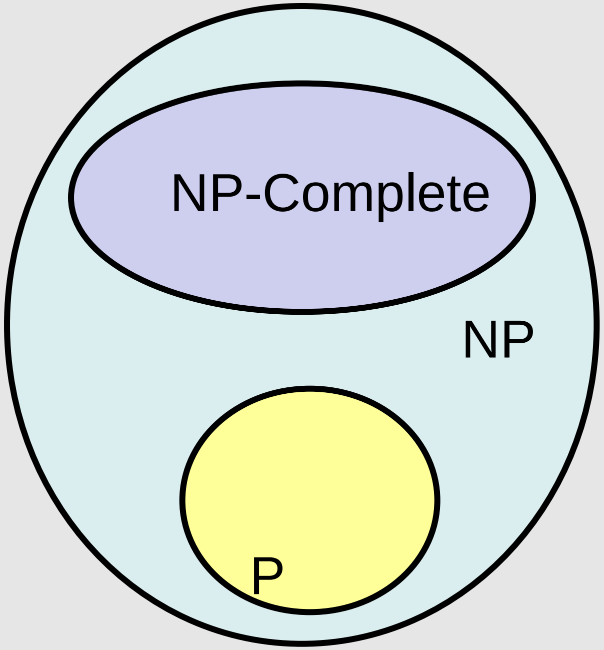
I can't find an efficient algorithm, but neither can all these famous people.

Algorithms vs. Lower bounds

- Algorithmic Theory
  - What we can compute
    - I can solve problem X with resources R
  - Proofs are almost always to give an algorithm that meets the resource bounds
- Lower bounds
  - How do we show that something can’t be done?

Theory of NP Completeness

The Universe



Polynomial Time

- P: Class of problems that can be solved in polynomial time
  - Corresponds with problems that can be solved efficiently in practice
  - Right class to work with “theoretically”

Decision Problems

- Theory developed in terms of yes/no problems
  - Independent set
    - Given a graph G and an integer K, does G have an independent set of size at least K
  - Shortest Path
    - Given a graph G with edge lengths, a start vertex s, and end vertex t, and an integer K, does the graph have a path between s and t of length at most K

What is NP?

- Problems solvable in non-deterministic polynomial time . . .

- Problems where “yes” instances have polynomial time checkable certificates

Certificate examples

- Independent set of size K
  - The Independent Set
- Satifisfiable formula
  - Truth assignment to the variables
- Hamiltonian Circuit Problem
  - A cycle including all of the vertices
- K-coloring a graph
  - Assignment of colors to the vertices

Certifiers and Certificates: 3-Satisfiability

SAT: Does a given CNF formula have a satisfying formula

Certificate: An assignment of truth values to the n boolean variables

Certifier: Check that each clause has at least one true literal,

instance s

$$\left(\overline{x_1} \lor x_2 \lor x_3\right) \land \left(x_1 \lor \overline{x_2} \lor x_3\right) \land \left(x_1 \lor x_2 \lor x_4\right) \land \left(\overline{x_1} \lor \overline{x_3} \lor \overline{x_4}\right)$$

certificate t

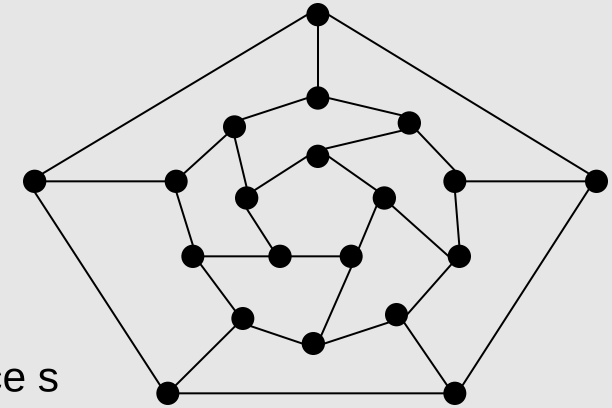
$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

Certifiers and Certificates: Hamiltonian Cycle

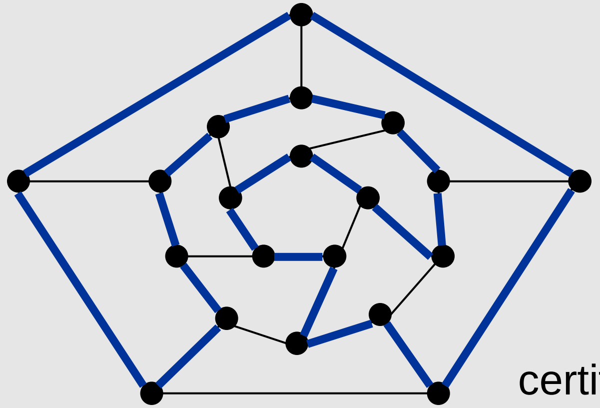
HAM-CYCLE. Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?

Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.



instance s



certificate t

Polynomial time reductions

- Y is Polynomial Time Reducible to X
  - Solve problem Y with a polynomial number of computation steps and a polynomial number of calls to a black box that solves X
  - Notations: $Y <_P X$
- Usually, this is converting an input of Y to an input for X, solving X, and then converting the answer back

Composability Lemma

- If $X <_P Y$ and $Y <_P Z$ then $X <_P Z$

Lemmas

- Suppose $Y <_P X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.

- Suppose $Y <_P X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

NP-Completeness

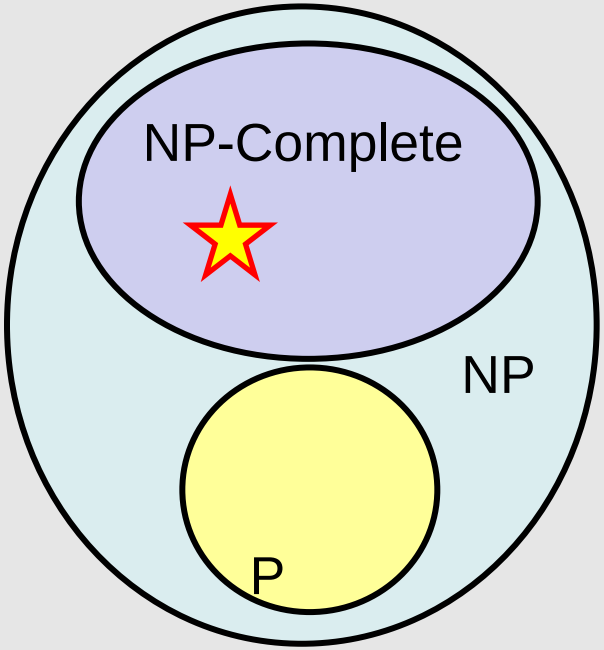
- A problem X is NP-complete if
  - X is in NP
  - For every Y in NP, $Y <_P X$

- X is a “hardest” problem in NP

- If X is NP-Complete, Z is in NP and $X <_P Z$
  - Then Z is NP-Complete

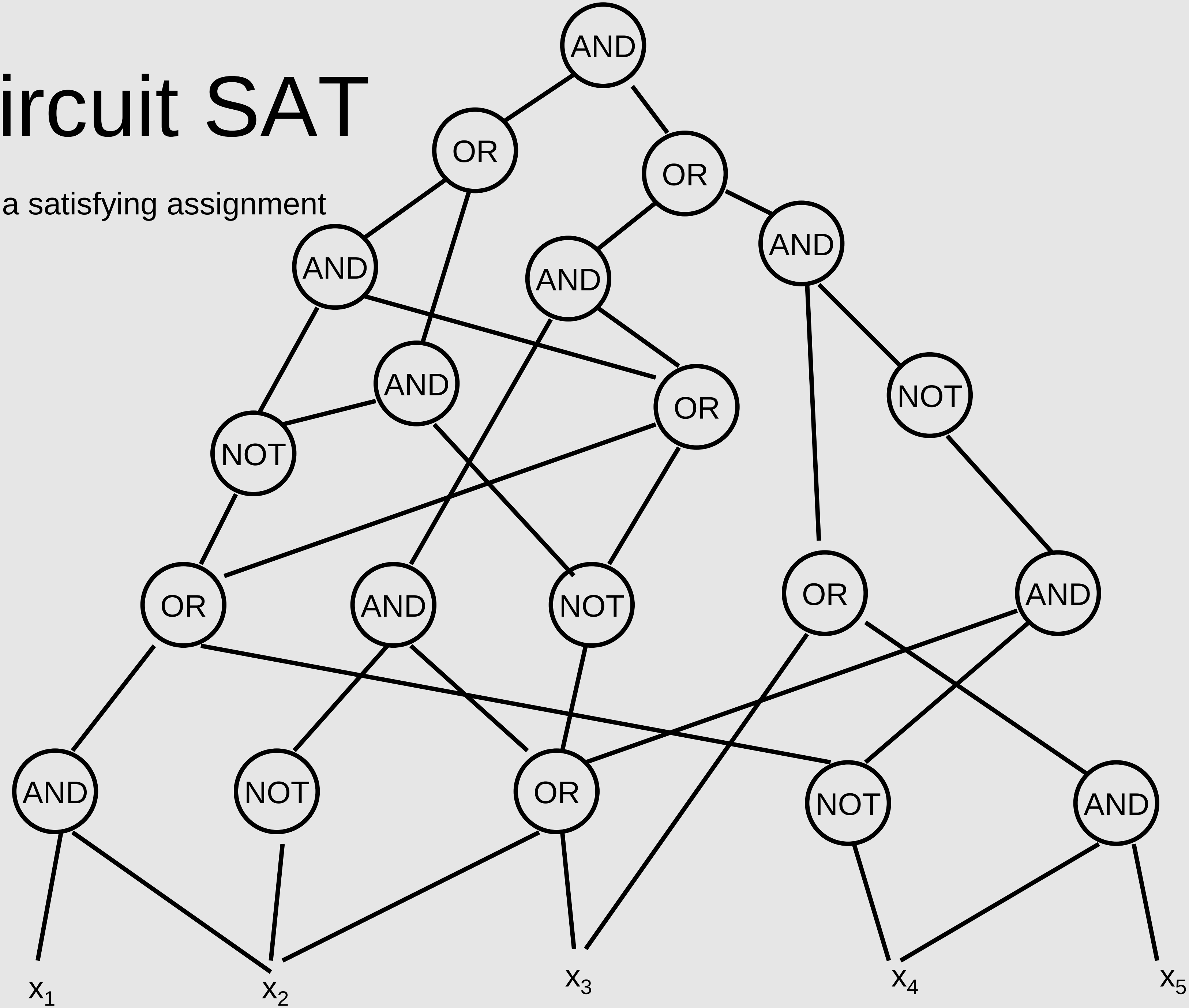
Cook’s Theorem

- There is an NP Complete problem
  - The Circuit Satisfiability Problem

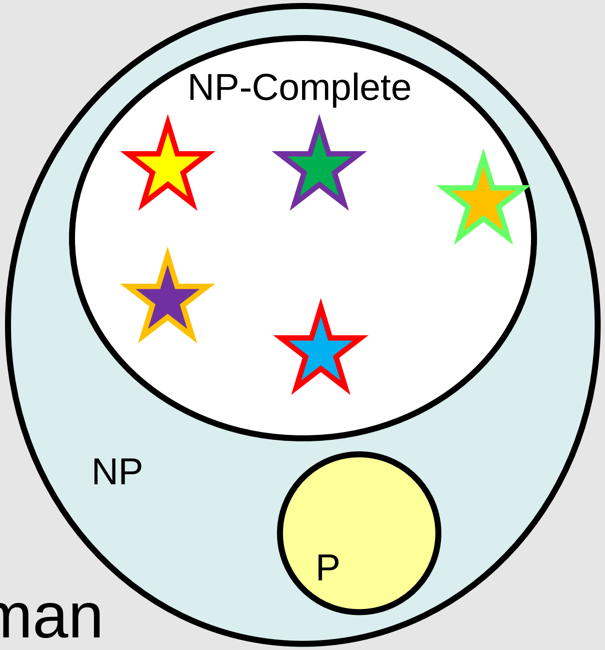


Circuit SAT

Find a satisfying assignment



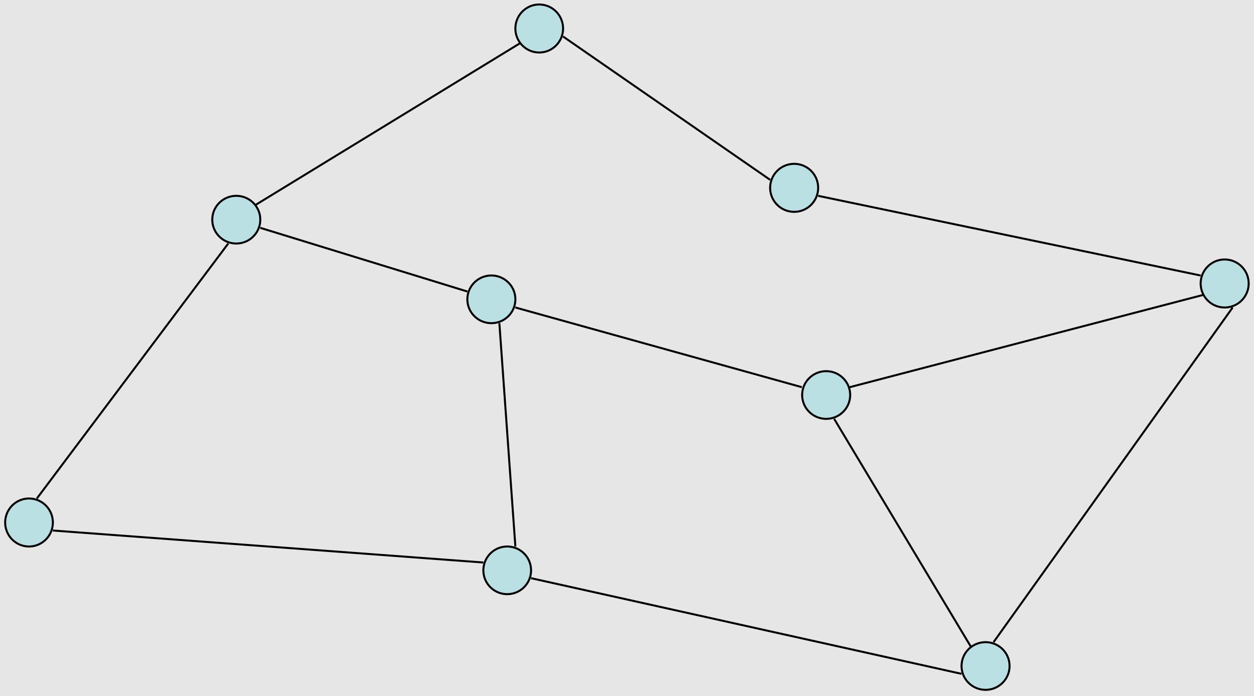
Populating the NP-Completeness Universe



- Circuit Sat $<_P$ 3-SAT
- 3-SAT $<_P$ Independent Set
- 3-SAT $<_P$ Vertex Cover
- Independent Set $<_P$ Clique
- 3-SAT $<_P$ Hamiltonian Circuit
- Hamiltonian Circuit $<_P$ Traveling Salesman
- 3-SAT $<_P$ Integer Linear Programming
- 3-SAT $<_P$ Graph Coloring
- 3-SAT $<_P$ Subset Sum
- Subset Sum $<_P$ Scheduling with Release times and deadlines

Graph Coloring

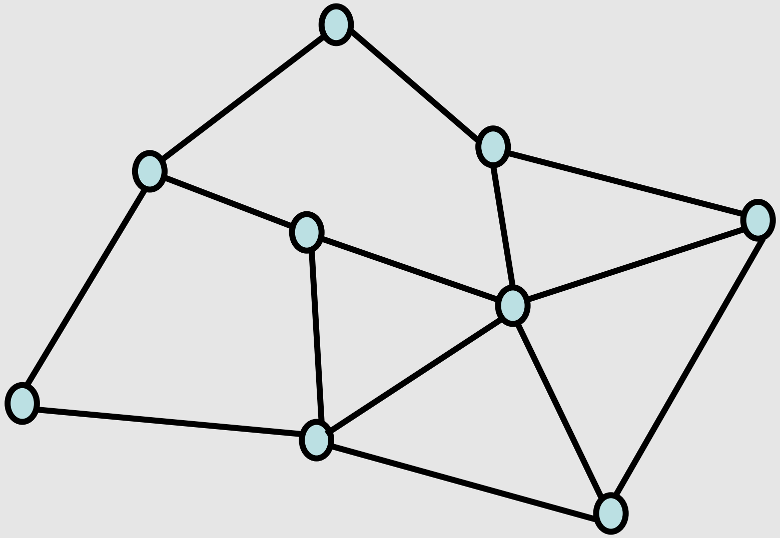
- NP-Complete
  - Graph 3-coloring
- Polynomial
  - Graph 2-Coloring

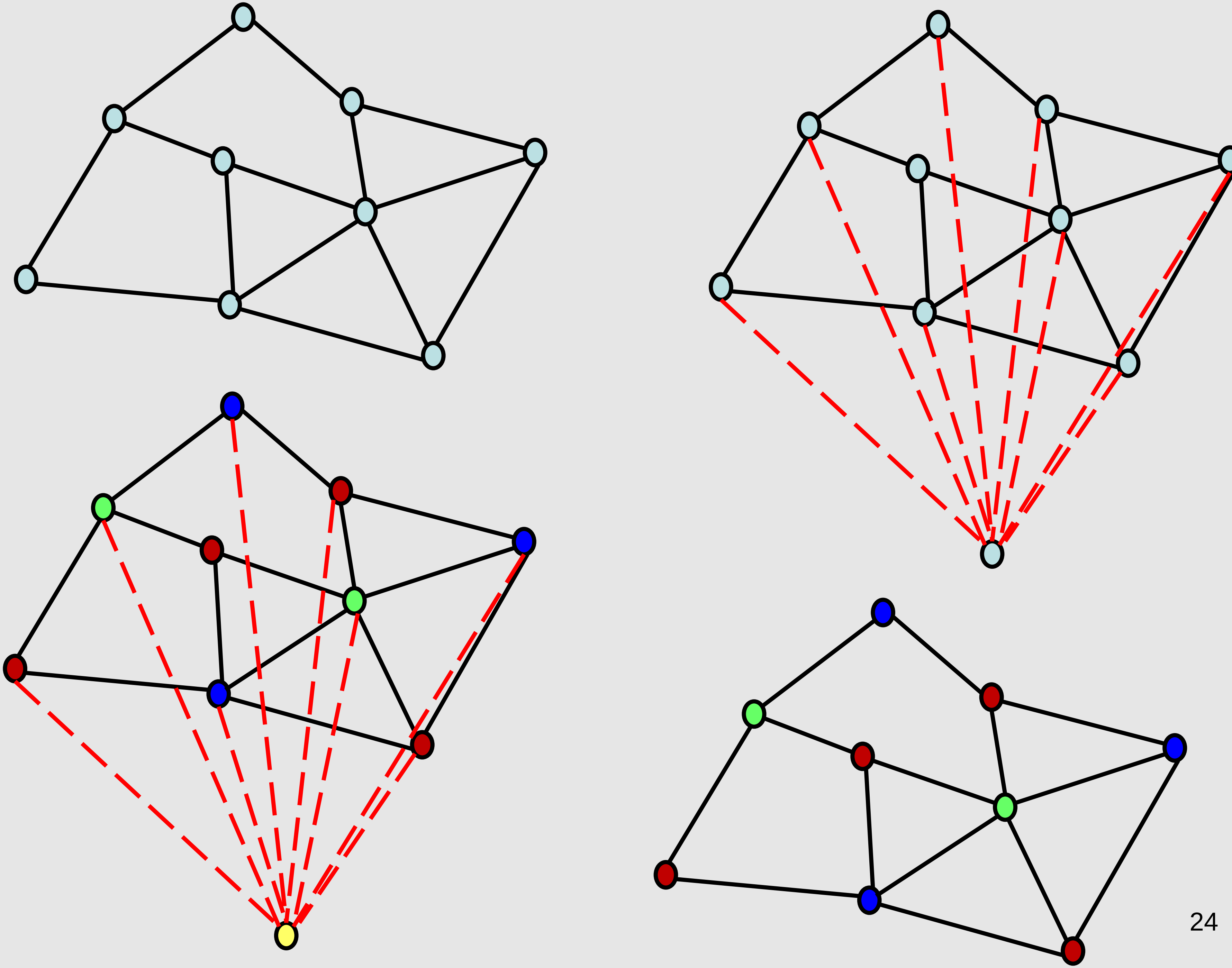


Graph 4-Coloring

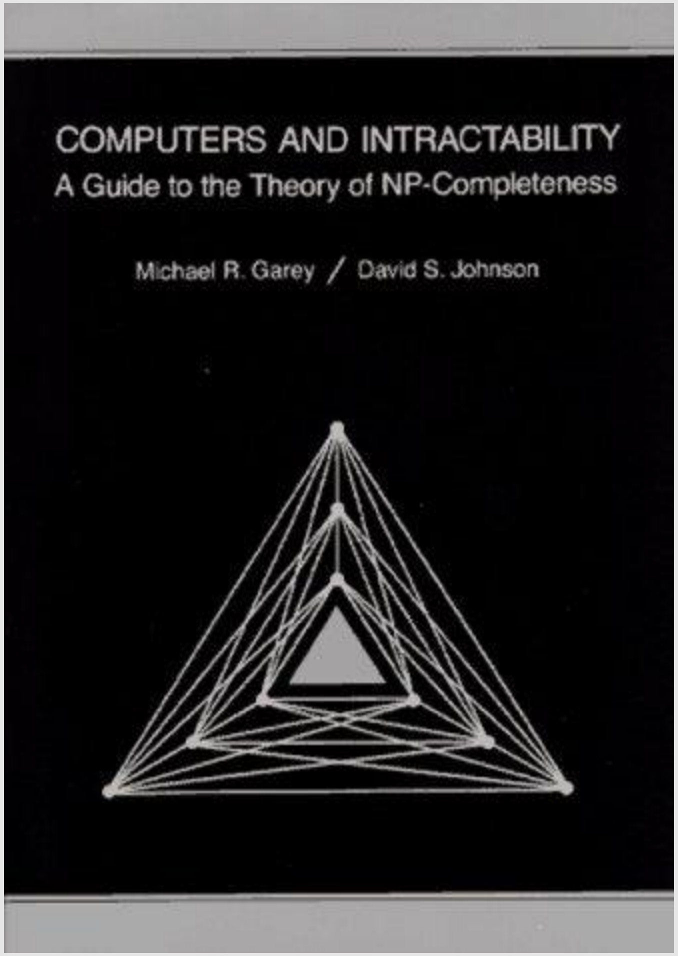
- Given a graph G, can G be colored with 4 colors?
- Prove 4-Coloring is NP Complete
- Proof: 3-Coloring $<_P$ 4-Coloring
- Show that you can 3-Color a graph if you have an algorithm to 4-Color a graph

3-Coloring $<_P$ 4-Coloring





Garey and Johnson



P vs. NP Question