

CSE 417
Algorithms and Complexity

Autumn 2024

Lecture 26

Network Flow Applications

Announcements

- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, December 9, 8:30 AM
  - – One Hour Fifty Minutes

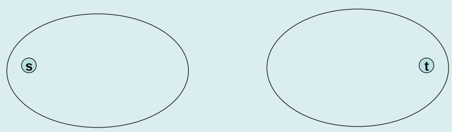
Wed, Nov 26	Net Flow Applications
Mon, Dec 2	Net Flow Applications + NP-Completeness
Wed, Dec 4	NP-Completeness
Fri, Dec 6	NP-Completeness
Mon, Dec 9	Final Exam

Outline

- ~~Network flow definitions~~
- ~~Flow examples~~
- ~~Augmenting Paths~~
- ~~Residual Graph~~
- ~~Ford Fulkerson Algorithm~~
- ~~Cuts~~
- ~~Maxflow-MinCut Theorem~~
- ~~Maxflow Algorithms~~
- Simple applications of Max Flow
- Non-simple applications of Max Flow

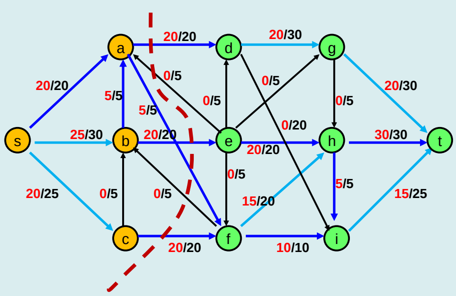
Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Problem: Find the s-t Cut with minimum capacity



Review

Max Flow / Min Cut



Max Flow - Min Cut Theorem

- There exists a cut S, T such that
  Flow(S,T) = Cap(S,T)
- Proof also shows that Ford Fulkerson algorithm finds a maximum flow

Ford Fulkerson Runtime

- Cost per phase X number of phases
- Phases
  - Capacity leaving source: C
  - Add at least one unit per phase
- Cost per phase
  - Build residual graph: O(m)
  - Find s-t path in residual: O(m)

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Network flow performance

- Ford-Fulkerson algorithm
  - O(mC)
- Find the maximum capacity augmenting path
  - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - O(mnlog n) time algorithm for network flow
- Interior Point Methods
  - O(m + n)

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Problem Reduction

- Reduce Problem A to Problem B
  - Convert an instance of Problem A to an instance of Problem B
  - Use a solution of Problem B to get a solution to Problem A

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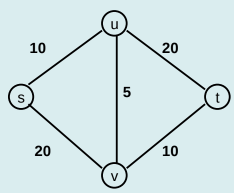
Undirected Network Flow

- Undirected graph with edge capacities
- Flow may go either direction along the edges (subject to the capacity constraints)



Construct an equivalent flow problem

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Bipartite Matching

- A graph G=(V,E) is bipartite if the vertices can be partitioned into disjoints sets X,Y
- A matching M is a subset of the edges that does not share any vertices
- Find a matching as large as possible

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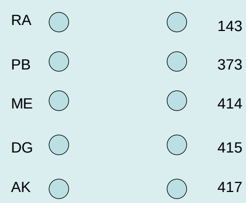
Application

- A collection of teachers
- A collection of courses
- And a graph showing which teachers can teach which courses



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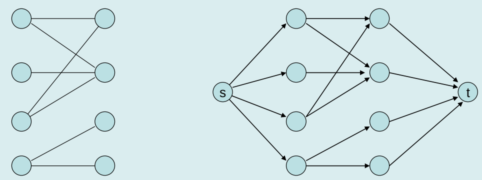
Converting Matching to Network Flow



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Multi-source network flow

- Multi-source network flow
  - Sources $s_1, s_2, \ldots, s_k$
  - Sinks $t_1, t_2, \ldots, t_j$
- Solve with Single source network flow

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Resource Allocation: Assignment of reviewers

- A set of papers $P_1, \ldots, P_n$
- A set of reviewers $R_1, \ldots, R_m$
- Paper P_i requires A_i reviewers
- Reviewer R_j can review B_j papers
- For each reviewer R_j, there is a list of paper $L_{j1}, \ldots, L_{jk}$ that R_j is qualified to review

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Baseball elimination

- Can the Dinosaurs win the league?
- Remaining games:
  - AB, AC, AD, AD, AD, BC, BC, BC, BD, CD

	W	L
Ants	4	2
Bees	4	2
Cockroaches	3	3
Dinosaurs	1	5

A team wins the league if it has strictly more wins than any other team at the end of the season
A team ties for first place if no team has more wins, and there is some other team with the same number of wins

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Baseball elimination

- Can the Fruit Flies win or tie the league?
- Remaining games:
  - AC, AD, AD, AD, AF, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, BF, CE, CE, CE, CF, CF, DE, DF, EF, EF

	W	L
Ants	17	12
Bees	16	7
Cockroaches	16	7
Dinosaurs	14	13
Earthworms	14	10
Fruit Flies	12	15

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Assume Fruit Flies win remaining games

- Fruit Flies are tied for first place if no team wins more than 19 games
- Allowable wins
  - Ants (2)
  - Bees (3)
  - Cockroaches (3)
  - Dinosaurs (5)
  - Earthworms (5)
- 18 games to play
  - AC, AD, AD, AD, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, CE, CE, CE, DE

	W	L
Ants	17	13
Bees	16	8
Cockroaches	16	9
Dinosaurs	14	14
Earthworms	14	12
Fruit Flies	19	15

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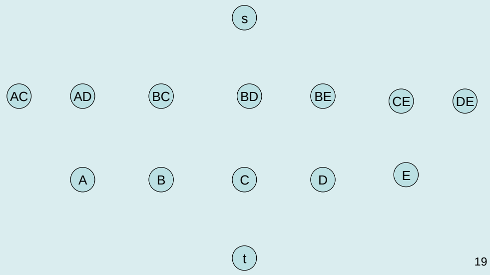
Remaining games

AC, AD, AD, AD, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, CE, CE, CE, DE



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Minimum Cut Applications

- Image Segmentation
- Open Pit Mining / Task Selection Problem
- Reduction to Min Cut problem

S, T is a cut if S, T is a partition of the vertices with
s in S and t in T
The capacity of an S, T cut is the sum of the capacities of all edges going from S to T

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Image Segmentation

- Separate foreground from background

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Image analysis

- a_i: value of assigning pixel i to the foreground
- b_i: value of assigning pixel i to the background
- p_{ij}: penalty for assigning i to the foreground, j to the background or vice versa
- A: foreground, B: background
- $Q(A,B) = \Sigma_{\{i \text{ in } A\}} a_i + \Sigma_{\{j \text{ in } B\}} b_j - \Sigma_{\{(i,j) \text{ in } E,\ i \text{ in } A,\ j \text{ in } B\}} p_{ij}$

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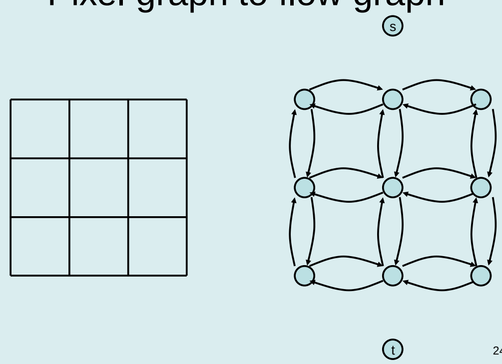
Pixel graph to flow graph



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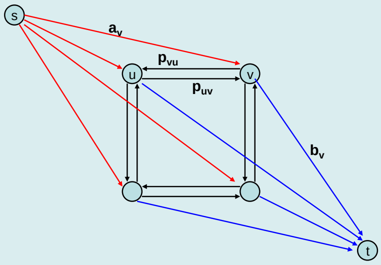
Mincut Construction

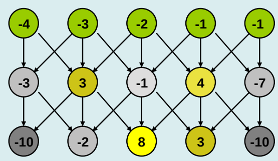


Open Pit Mining (Task selection)

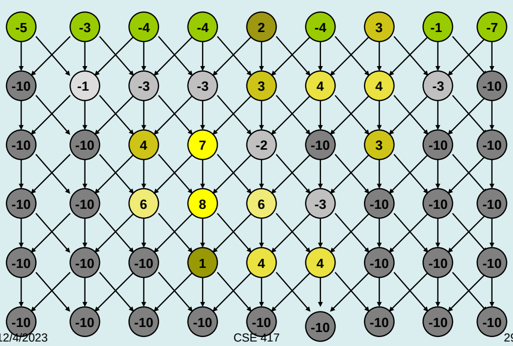
Open Pit Mining

- Each unit of earth has a profit (possibly negative)
- Getting to the ore below the surface requires removing the dirt above
- Test drilling gives reasonable estimates of costs
- Plan an optimal mining operation

Mine Graph

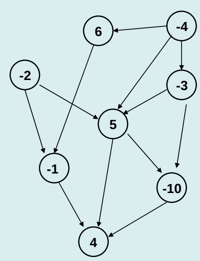


Determine an optimal mine



Generalization

- Precedence graph G=(V,E)
- Each v in V has a profit p(v)
- A set F is *feasible* if when w in F, and (v,w) in E, then v in F.
- Find a feasible set to maximize the profit

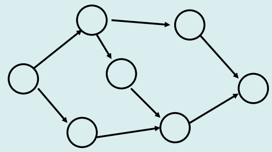


Min cut algorithm for profit maximization

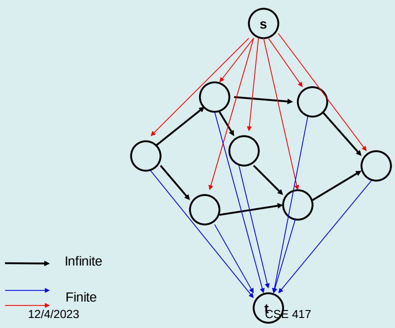
- Construct a flow graph where the minimum cut identifies a feasible set that maximizes profit

Precedence graph construction

- Precedence graph G=(V,E)
- Each edge in E has infinite capacity
- Add vertices s, t
- Each vertex in V is attached to s and t with finite capacity edges

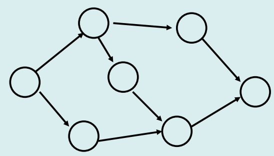


Find a finite value cut with at least two vertices on each side of the cut



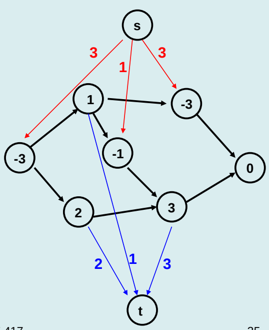
The sink side of a finite cut is a feasible set

- No edges permitted from S to T
- If a vertex is in T, all of its ancestors are in T

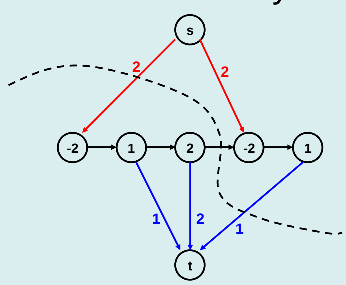


Setting the costs

- If p(v) > 0,
  - cap(v,t) = p(v)
  - cap(s,v) = 0
- If p(v) < 0
  - cap(s,v) = -p(v)
  - cap(v,t) = 0
- If p(v) = 0
  - cap(s,v) = 0
  - cap(v,t) = 0



Minimum cut gives optimal solution Why?

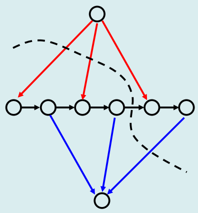


Computing the Profit

- Cost(W) = $\Sigma_{\{w \text{ in } W;\, p(w) < 0\}} -p(w)$
- Benefit(W) = $\Sigma_{\{w \text{ in } W;\, p(w) > 0\}}\ p(w)$
- Profit(W) = Benefit(W) – Cost(W)

- Maximum cost and benefit
  - C = Cost(V)
  - B = Benefit(V)

Express Cap(S,T) in terms of B, C, Cost(T), Benefit(T), and Profit(T)



Cap(S,T) = Cost(T) + Ben(S) = Cost(T) + Ben(S) + Ben(T) – Ben(T)
= B + Cost(T) – Ben(T) = B – Profit(T)