CSE 417
Algorithms and Complexity

Autumn 2024

Lecture 26

Network Flow Applications

Announcements

- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, December 9, 8:30 AM
  - One Hour Fifty Minutes

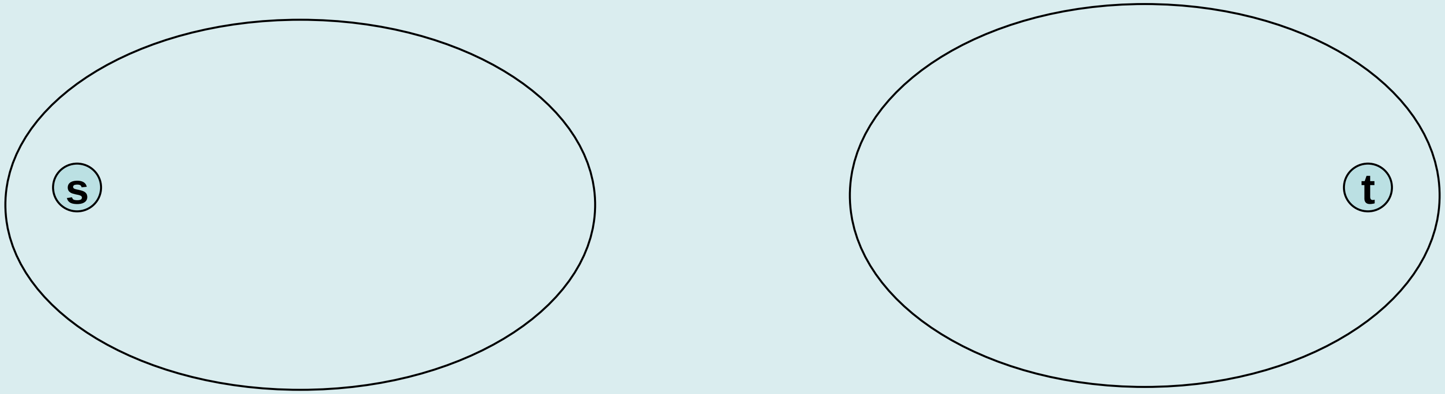
Wed, Nov 26	Net Flow Applications
Mon, Dec 2	Net Flow Applications + NP-Completeness
Wed, Dec 4	NP-Completeness
Fri, Dec 6	NP-Completeness
Mon, Dec 9	Final Exam

Outline

- ~~Network flow definitions~~
- ~~Flow examples~~
- ~~Augmenting Paths~~
- ~~Residual Graph~~
- ~~Ford Fulkerson Algorithm~~
- ~~Cuts~~
- ~~Maxflow-MinCut Theorem~~
- ~~Maxflow Algorithms~~
- Simple applications of Max Flow
- Non-simple applications of Max Flow

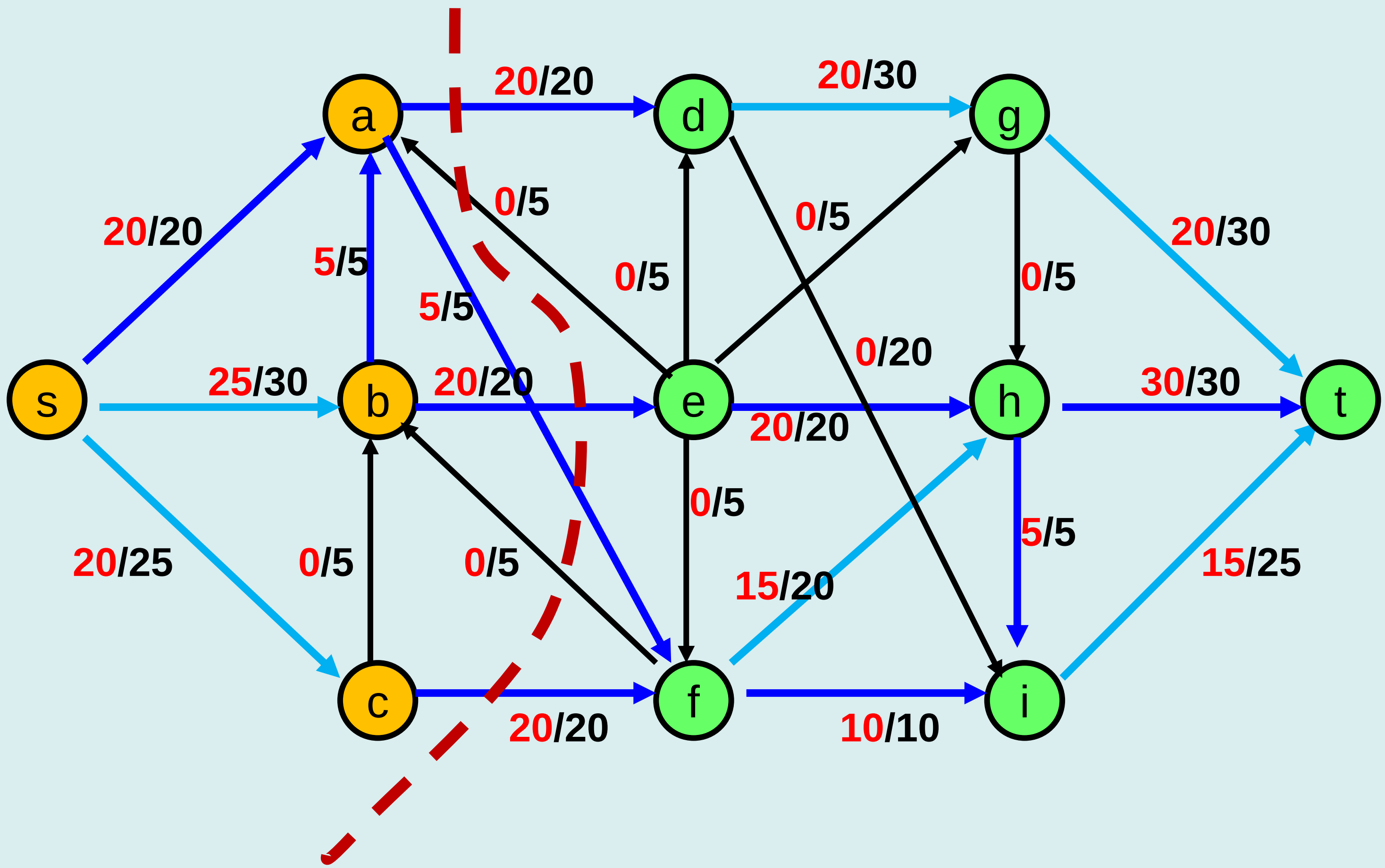
Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Problem: Find the s-t Cut with minimum capacity



Review

Max Flow / Min Cut



Max Flow - Min Cut Theorem

- There exists a cut S, T such that

$$\text{Flow}(S,T) = \text{Cap}(S,T)$$

- Proof also shows that Ford Fulkerson algorithm finds a maximum flow

Ford Fulkerson Runtime

- Cost per phase   X   number of phases

- Phases
  - Capacity leaving source: C
  - Add at least one unit per phase
- Cost per phase
  - Build residual graph: O(m)
  - Find s-t path in residual: O(m)

Network flow performance

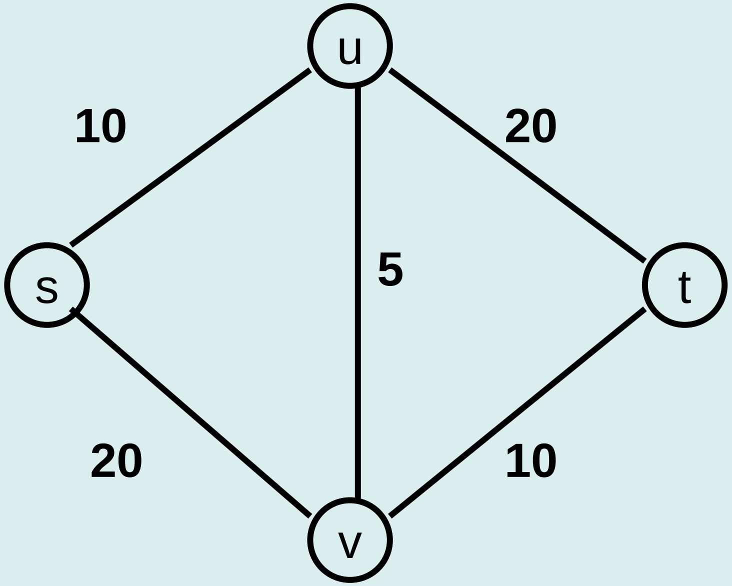
- Ford-Fulkerson algorithm
  - $O(mC)$
- Find the maximum capacity augmenting path
  - $O(m^2\log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn\log n)$ time algorithm for network flow
- Interior Point Methods
  - $O(m + n)$

Problem Reduction

- Reduce Problem A to Problem B
  - Convert an instance of Problem A to an instance of Problem B
  - Use a solution of Problem B to get a solution to Problem A

Undirected Network Flow

- Undirected graph with edge capacities
- Flow may go either direction along the edges (subject to the capacity constraints)



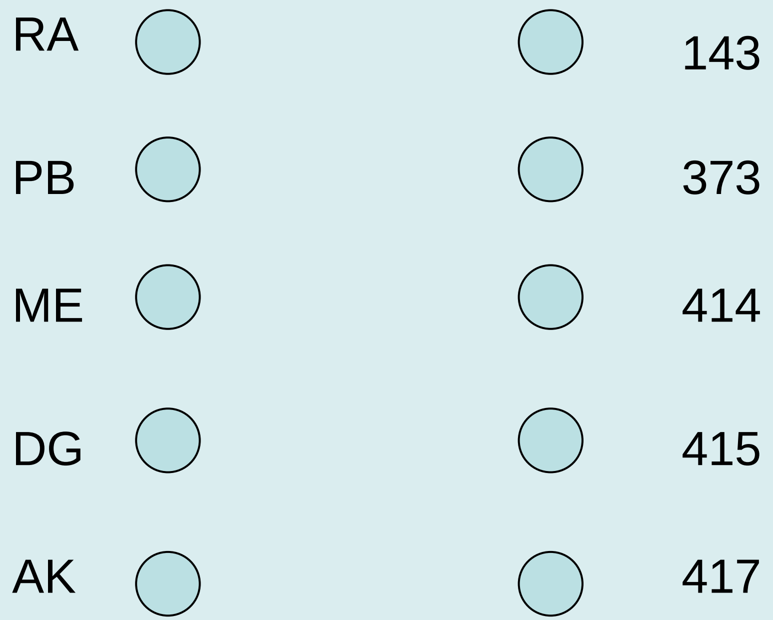
Construct an equivalent flow problem

Bipartite Matching

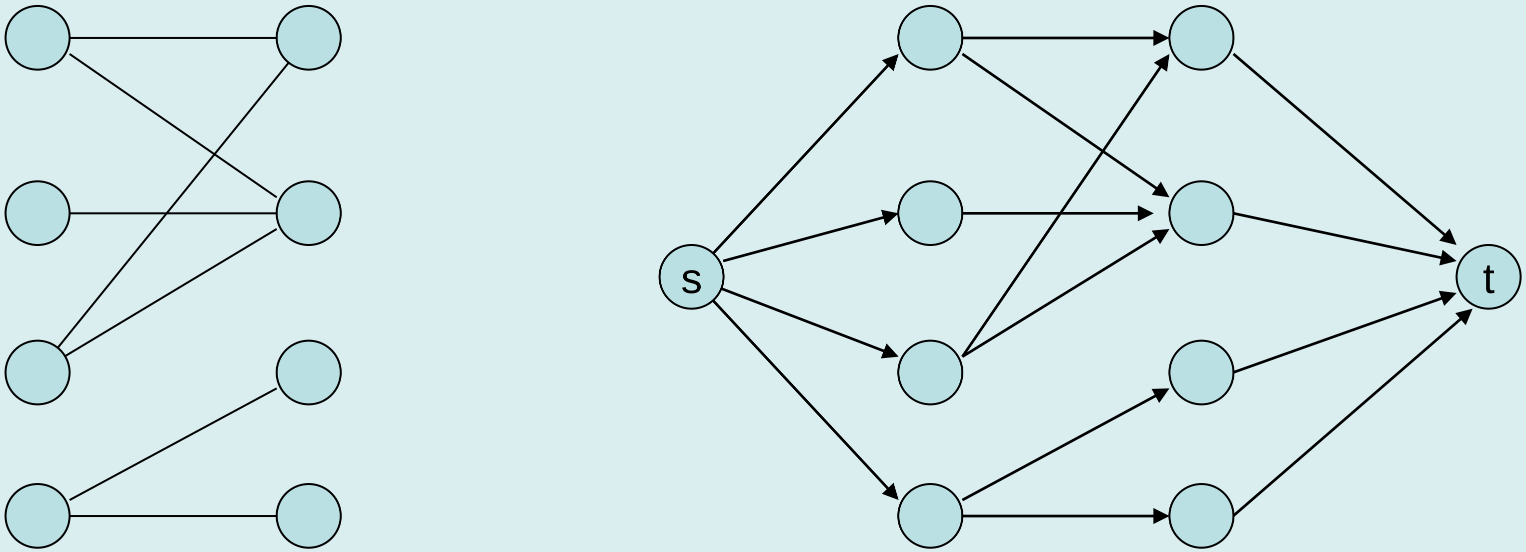
- A graph G=(V,E) is bipartite if the vertices can be partitioned into disjoints sets X,Y
- A matching M is a subset of the edges that does not share any vertices
- Find a matching as large as possible

Application

- A collection of teachers
- A collection of courses
- And a graph showing which teachers can teach which courses



Converting Matching to Network Flow



Multi-source network flow

- Multi-source network flow
  - Sources $s_1, s_2, . . ., s_k$
  - Sinks $t_1, t_2, . . . , t_j$
- Solve with Single source network flow

Resource Allocation: Assignment of reviewers

- A set of papers $P_1, \ldots, P_n$
- A set of reviewers $R_1, \ldots, R_m$
- Paper P_i requires A_i reviewers
- Reviewer R_j can review B_j papers
- For each reviewer R_j, there is a list of paper $L_{j1}, \ldots, L_{jk}$ that R_j is qualified to review

Baseball elimination

- Can the Dinosaurs win the league?
- Remaining games:
  - AB, AC, AD, AD, AD, BC, BC, BC, BD, CD

	W	L
Ants	4	2
Bees	4	2
Cockroaches	3	3
Dinosaurs	1	5

A team wins the league if it has strictly more wins than any other team at the end of the season
A team ties for first place if no team has more wins, and there is some other team with the same number of wins

Baseball elimination

- Can the Fruit Flies win or tie the league?
- Remaining games:
  - AC, AD, AD, AD, AF, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, BF, CE, CE, CE, CF, CF, DE, DF, EF, EF

	W	L
Ants	17	12
Bees	16	7
Cockroaches	16	7
Dinosaurs	14	13
Earthworms	14	10
Fruit Flies	12	15

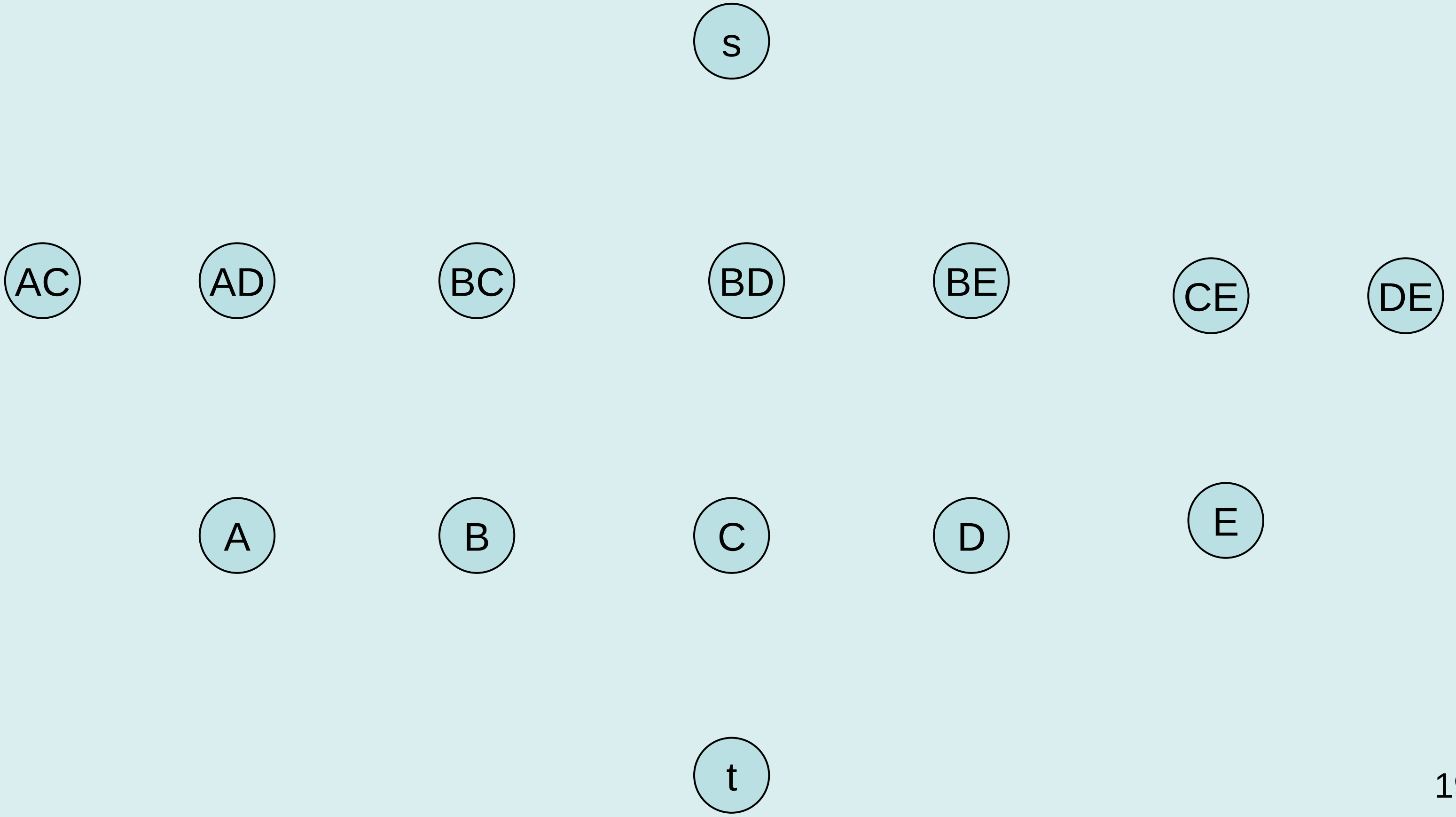
Assume Fruit Flies win remaining games

- Fruit Flies are tied for first place if no team wins more than 19 games
- Allowable wins
  - Ants (2)
  - Bees (3)
  - Cockroaches (3)
  - Dinosaurs (5)
  - Earthworms (5)
- 18 games to play
  - AC, AD, AD, AD, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, CE, CE, CE, DE

	W	L
Ants	17	13
Bees	16	8
Cockroaches	16	9
Dinosaurs	14	14
Earthworms	14	12
Fruit Flies	19	15

Remaining games

AC, AD, AD, AD, BC, BC, BC, BC, BC, BD, BE, BE, BE, BE, CE, CE, CE, DE



Minimum Cut Applications

- Image Segmentation
- Open Pit Mining / Task Selection Problem
- Reduction to Min Cut problem

S, T is a cut if S, T is a partition of the vertices with
s in S and t in T
The capacity of an S, T cut is the sum of the capacities of all edges going from S to T

Image Segmentation

- Separate foreground from background

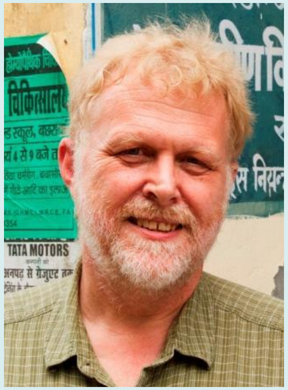
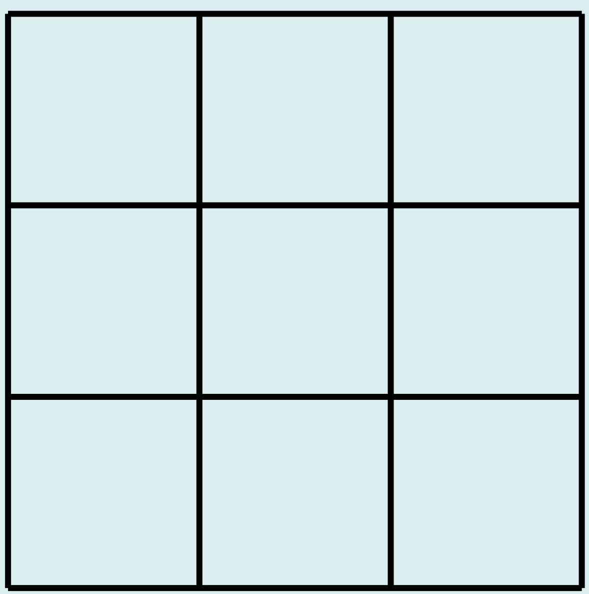
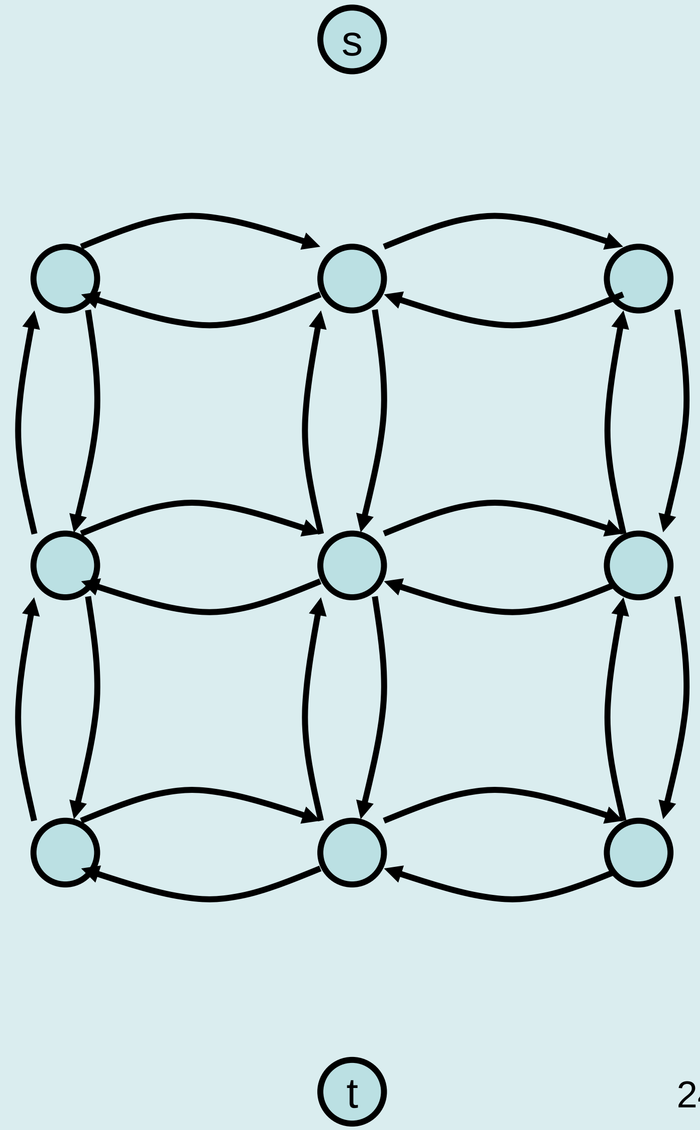


Image analysis

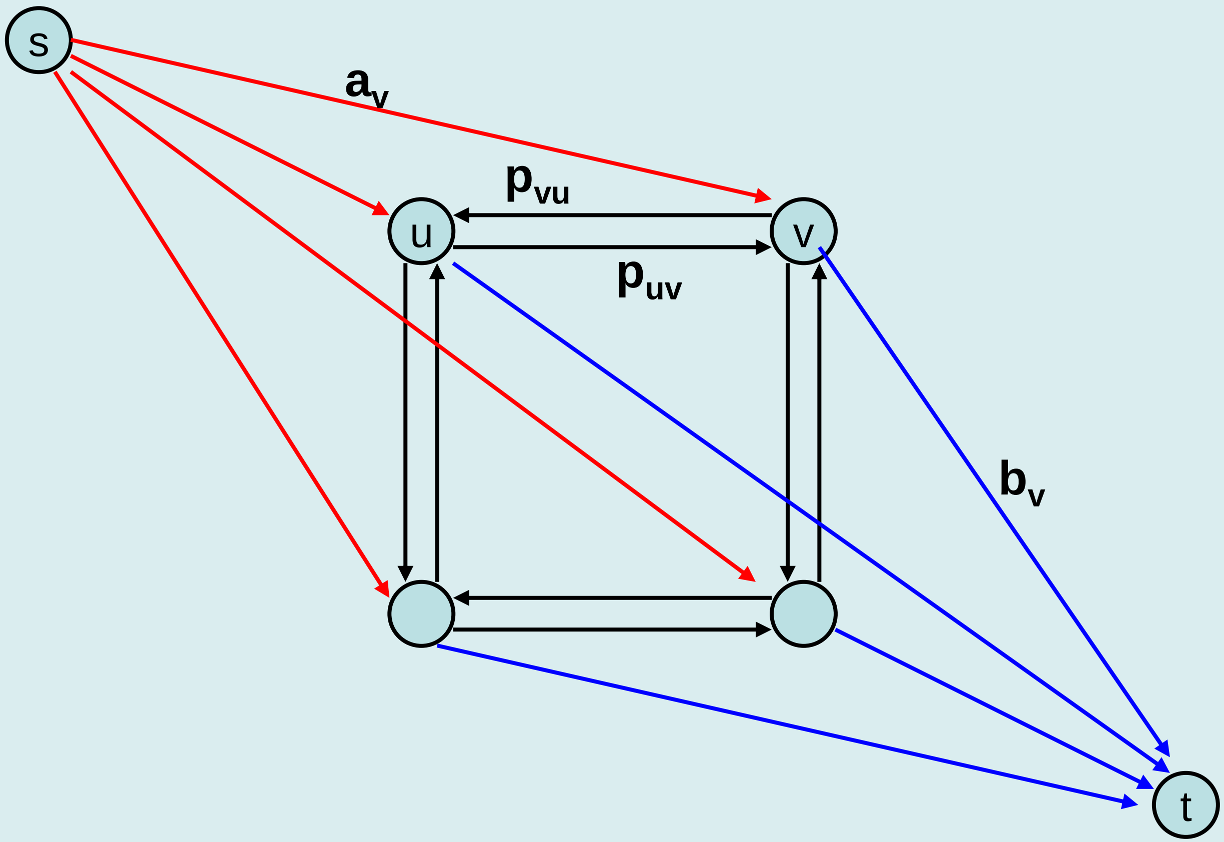
- a_i: value of assigning pixel i to the foreground
- b_i: value of assigning pixel i to the background
- p_{ij}: penalty for assigning i to the foreground, j to the background or vice versa
- A: foreground, B: background
- $Q(A,B) = \Sigma_{\{i \text{ in } A\}} a_i + \Sigma_{\{j \text{ in } B\}} b_j - \Sigma_{\{(i,j) \text{ in } E,\ i \text{ in } A,\ j \text{ in } B\}} p_{ij}$

Pixel graph to flow graph

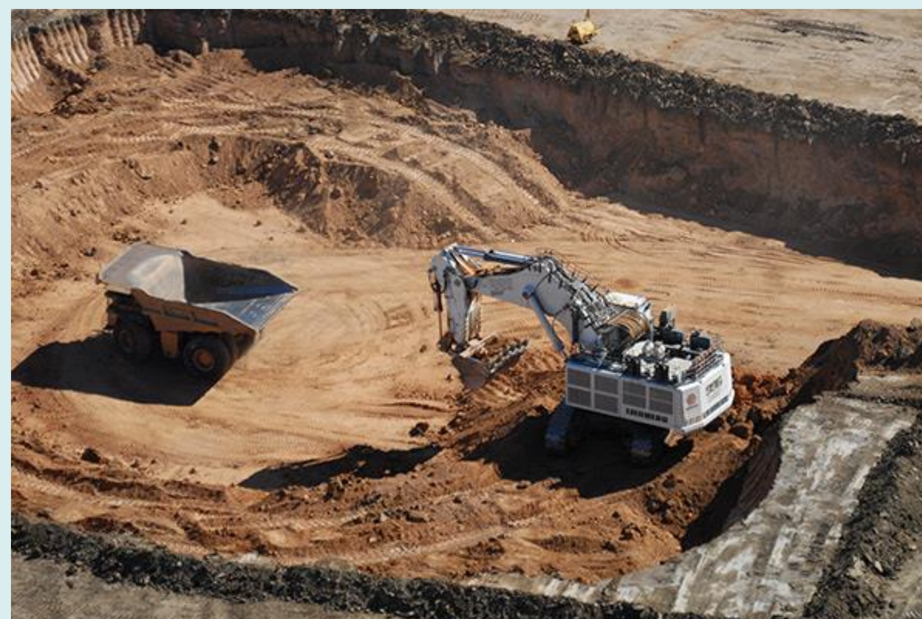




Mincut Construction



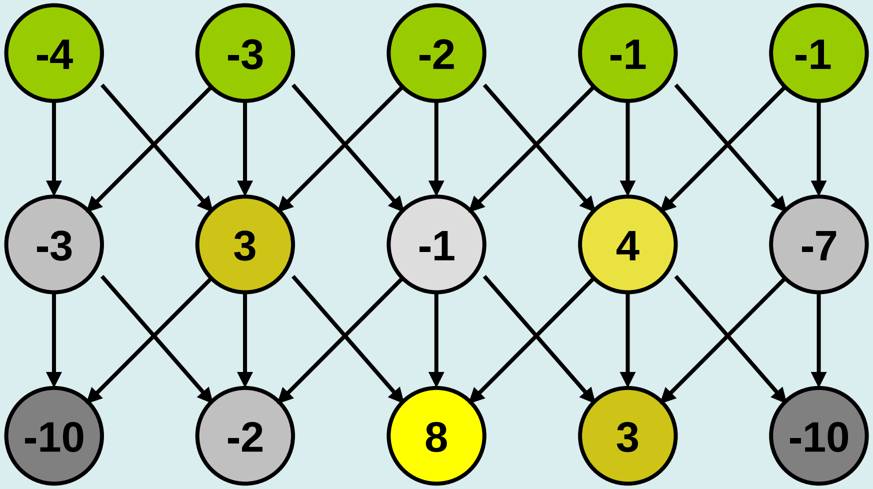
Open Pit Mining (Task selection)



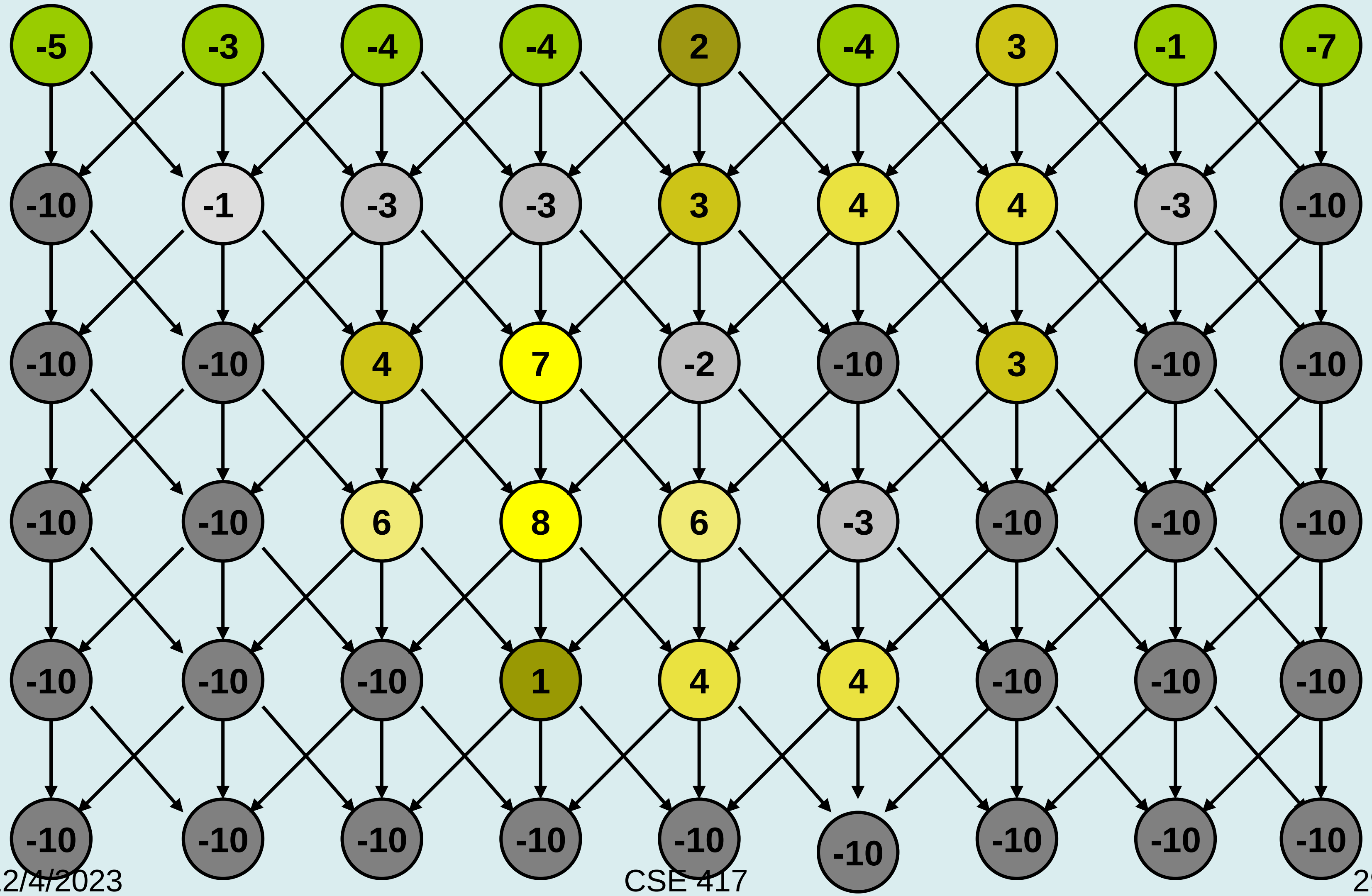
Open Pit Mining

- Each unit of earth has a profit (possibly negative)
- Getting to the ore below the surface requires removing the dirt above
- Test drilling gives reasonable estimates of costs
- Plan an optimal mining operation

Mine Graph

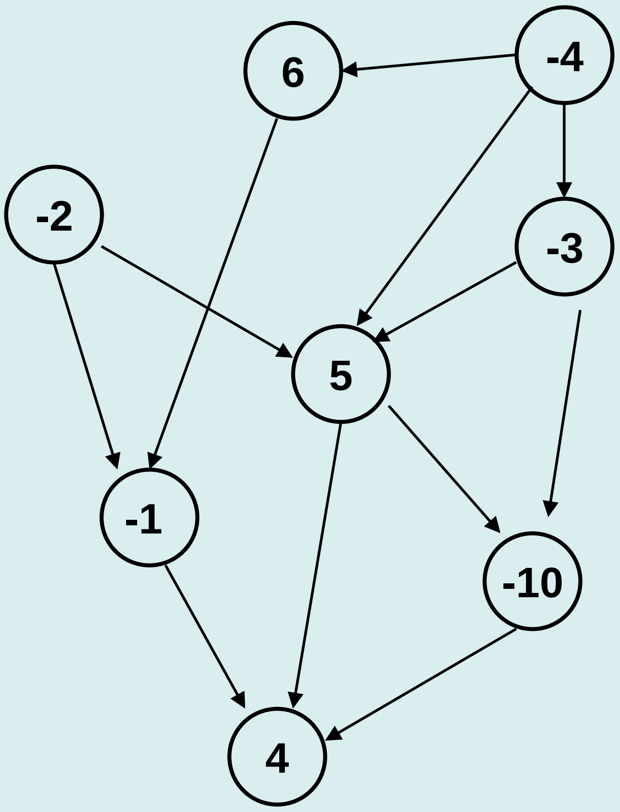


Determine an optimal mine



Generalization

- Precedence graph G=(V,E)
- Each v in V has a profit p(v)
- A set F is *feasible* if when w in F, and (v,w) in E, then v in F.
- Find a feasible set to maximize the profit

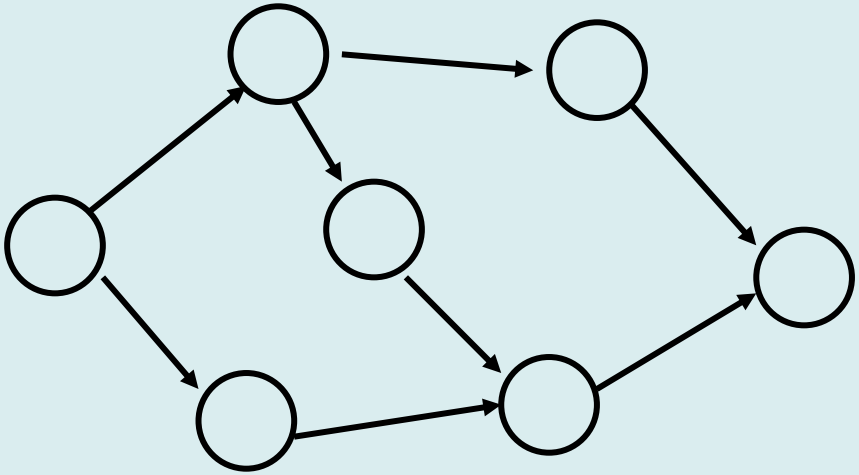


Min cut algorithm for profit maximization

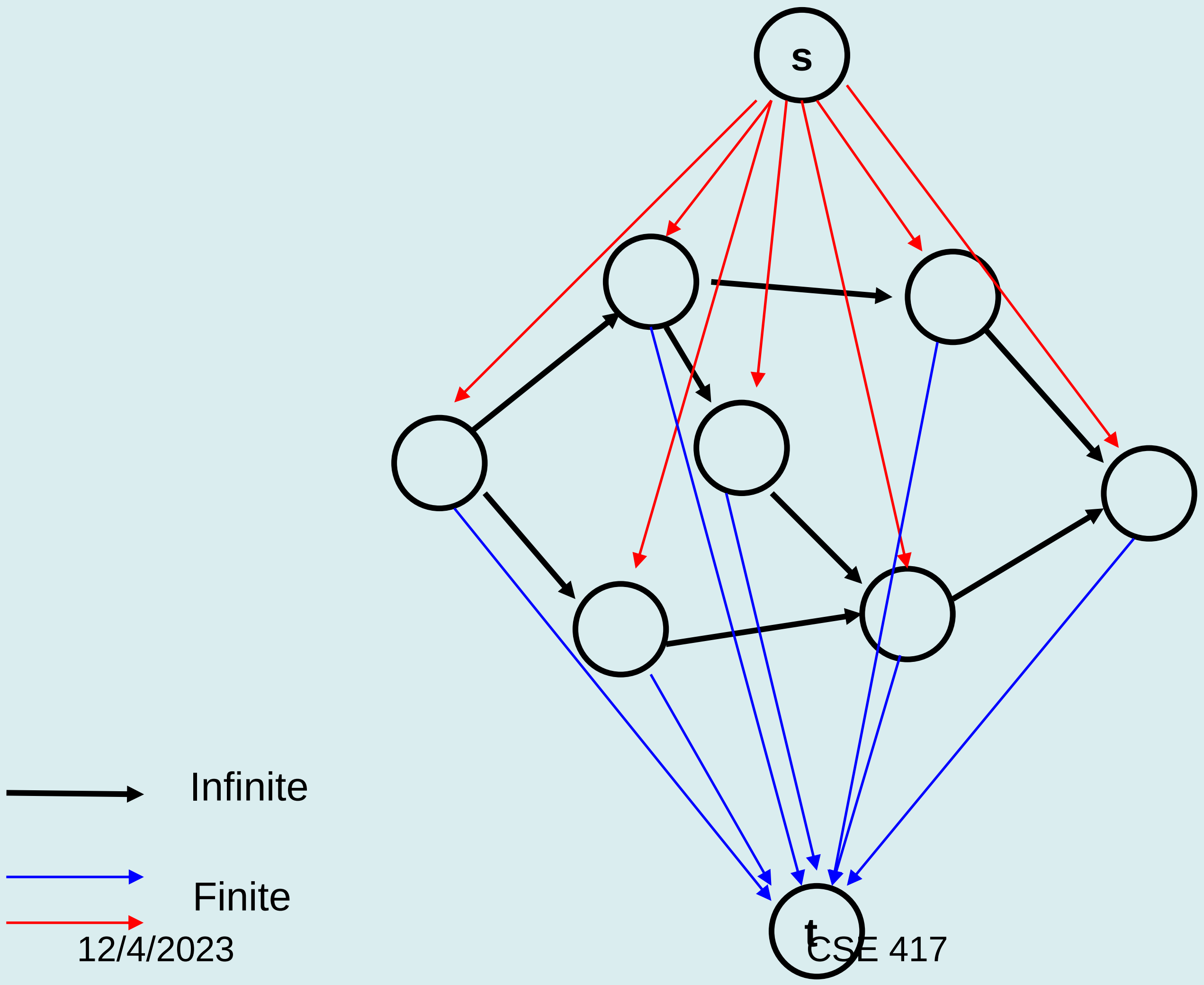
- Construct a flow graph where the minimum cut identifies a feasible set that maximizes profit

Precedence graph construction

- Precedence graph G=(V,E)
- Each edge in E has infinite capacity
- Add vertices s, t
- Each vertex in V is attached to s and t with finite capacity edges

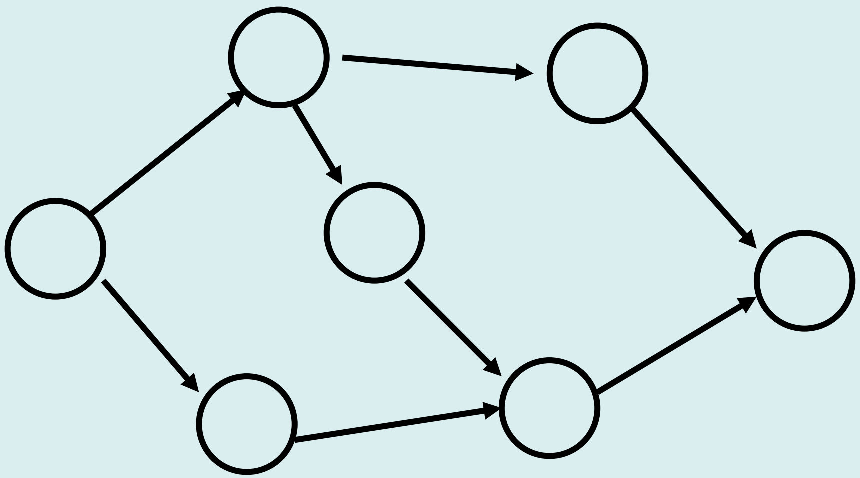


Find a finite value cut with at least two vertices on each side of the cut



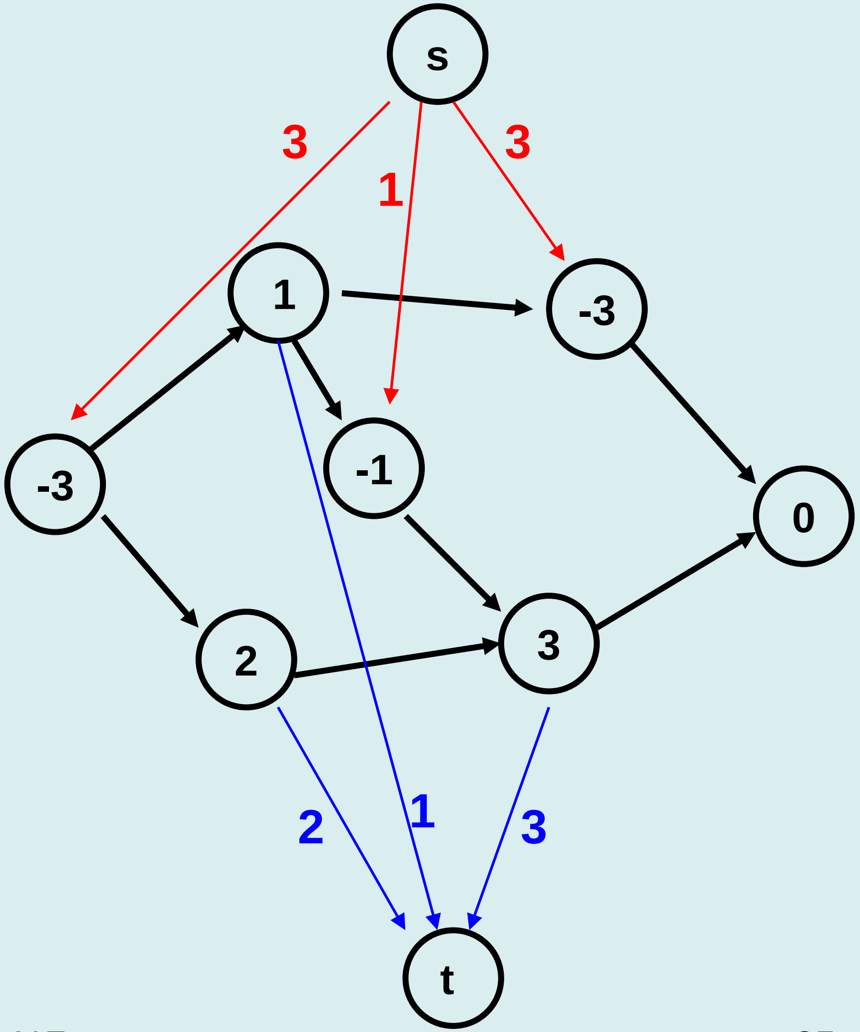
The sink side of a finite cut is a feasible set

- No edges permitted from S to T
- If a vertex is in T, all of its ancestors are in T

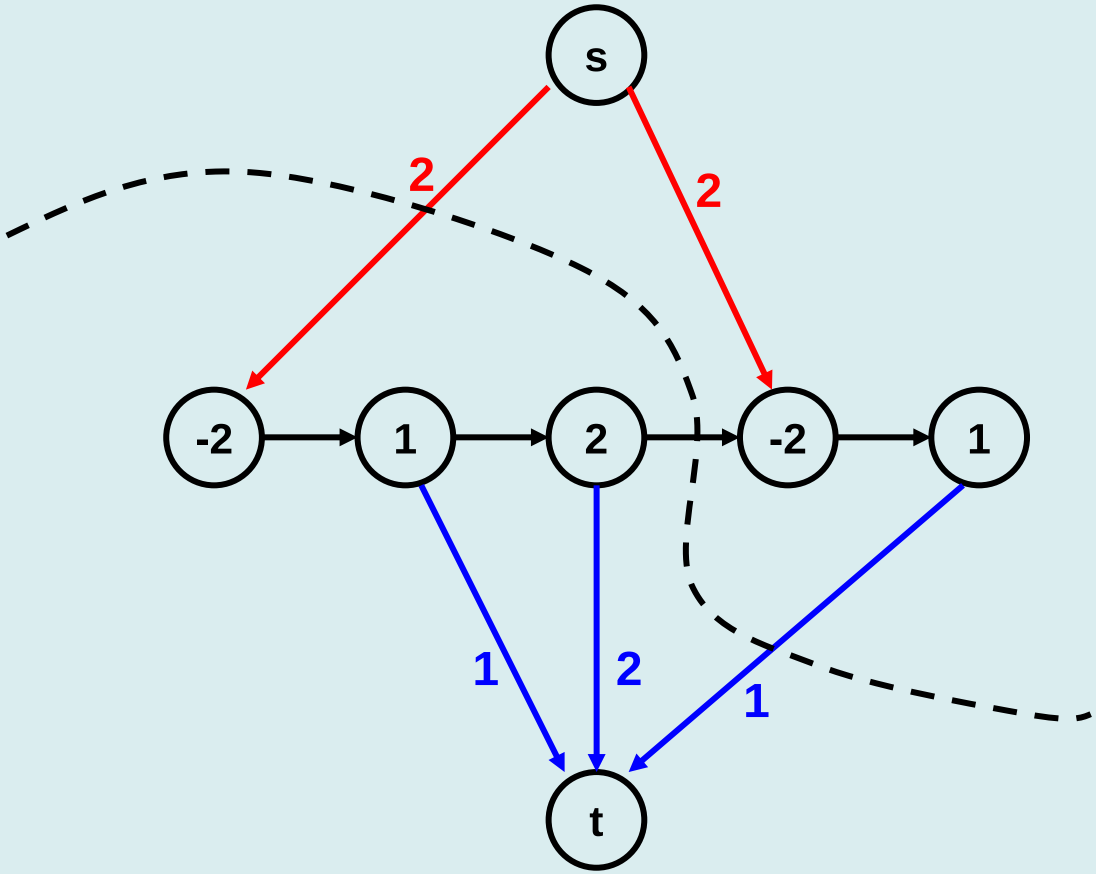


Setting the costs

- If p(v) > 0,
  - cap(v,t) = p(v)
  - cap(s,v) = 0
- If p(v) < 0
  - cap(s,v) = -p(v)
  - cap(v,t) = 0
- If p(v) = 0
  - cap(s,v) = 0
  - cap(v,t) = 0



Minimum cut gives optimal solution
Why?

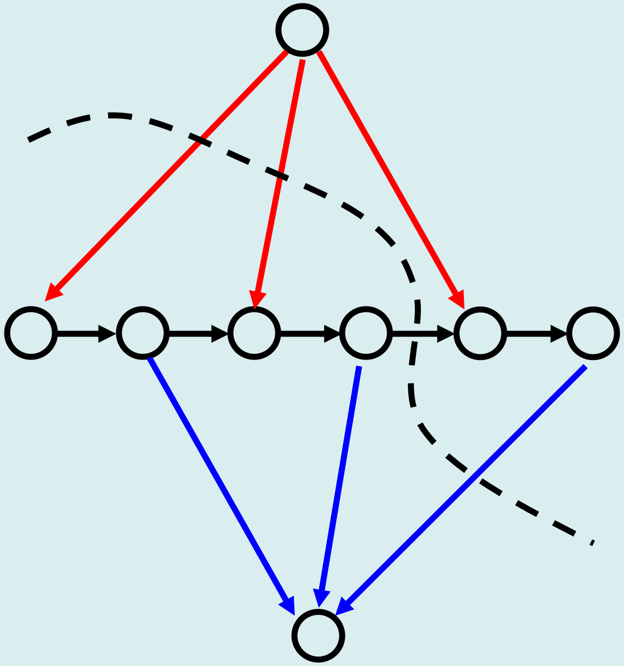


Computing the Profit

- $\text{Cost}(W) = \Sigma_{\{w \text{ in } W;\ p(w) < 0\}} -p(w)$
- $\text{Benefit}(W) = \Sigma_{\{w \text{ in } W;\ p(w) > 0\}}\ p(w)$
- Profit(W) = Benefit(W) – Cost(W)

- Maximum cost and benefit
  – C = Cost(V)
  – B = Benefit(V)

Express Cap(S,T) in terms of B, C, Cost(T), Benefit(T), and Profit(T)



Cap(S,T) = Cost(T) + Ben(S) = Cost(T) + Ben(S) + Ben(T) – Ben(T)

= B + Cost(T) – Ben(T) = B – Profit(T)