# Lecture25

# CSE 417
# Algorithms and Complexity

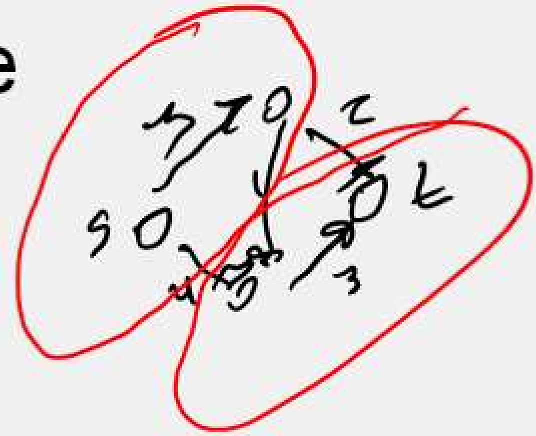
## Lecture 25

Autumn 2024

Network Flow, Part 2

1

# Outline



- ~~Network flow definitions~~
- ~~Flow examples~~
- ~~Augmenting Paths~~
- ~~Residual Graph~~
- ~~Ford Fulkerson Algorithm~~
- Cuts
- Maxflow-MinCut Theorem
- Simple applications of Max Flow

2

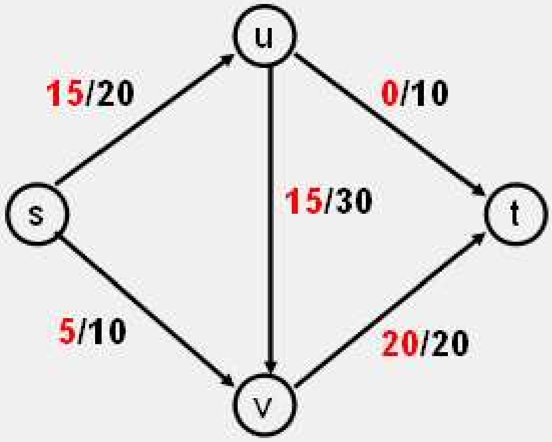
# Network Flow Definitions

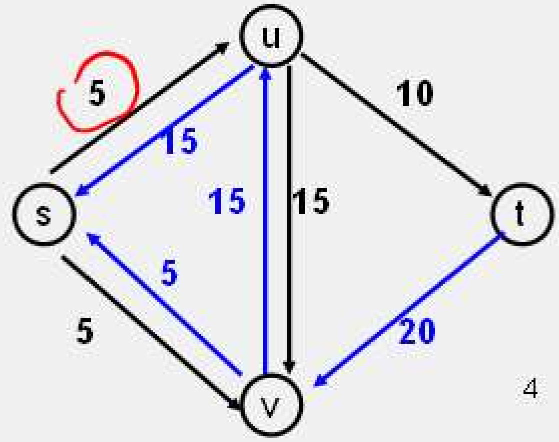
- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, $c(e) \geq 0$
- Problem, assign flows f(e) to the edges such that:
  - $0 \leq f(e) \leq c(e)$
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

3

# Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph $G_R$
  - G: edge e from u to v with capacity c and flow f
  - $G_R$: edge e' from u to v with capacity c – f
  - $G_R$: edge e'' from v to u with capacity f



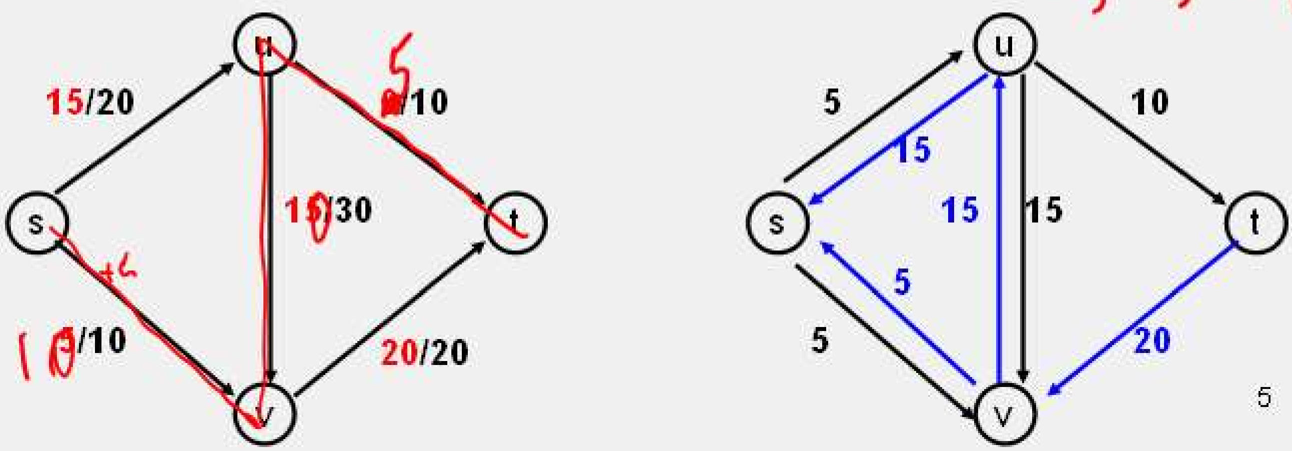




4

# Augmenting Path Algorithm

- Augmenting path in residual graph
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between $v_j$ and $v_{j+1}$ for j = 1 … k-1

s, v, u, t



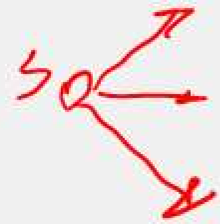

5

# Adding flow along a path in the residual graph

- Let P be an s-t path in the residual graph with capacity b
- b units of flow can be added along P in the graph G
- Need to show:
  - new flow satisfies capacity constraints
  - new flow satisfies conservation constraints

6

# Ford-Fulkerson Algorithm (1956)




while not done

  Construct residual graph $G_R$

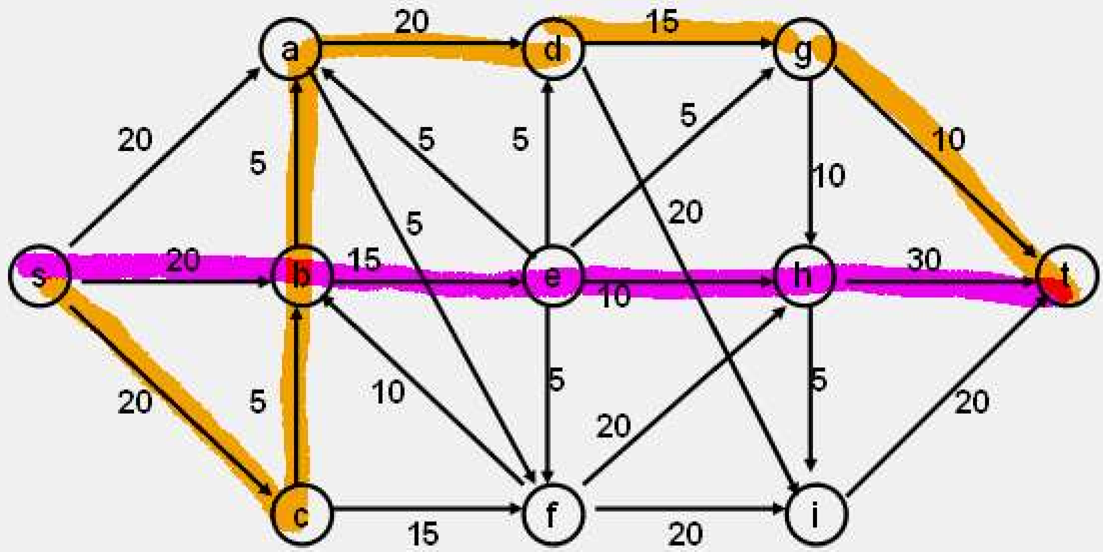
  Find an s-t path P in $G_R$ with capacity $b > 0$

  Add b units of flow along path P in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

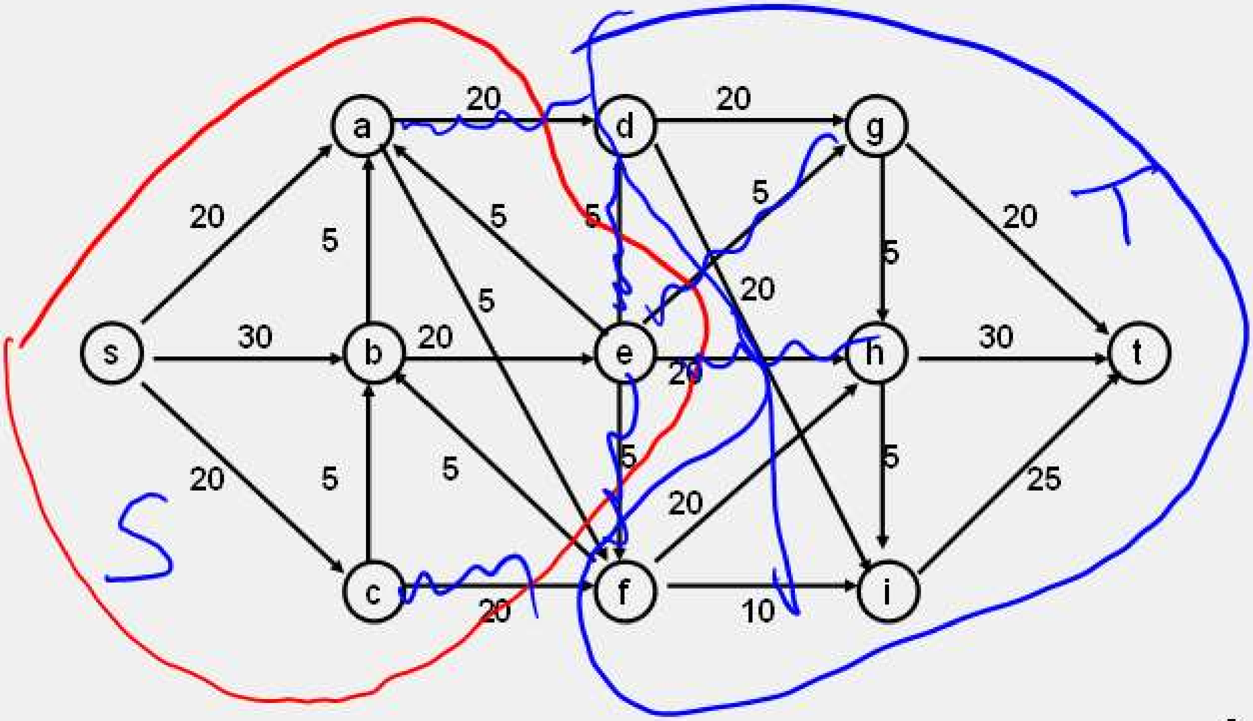
7

# Flow Example I




8

# Flow Example II




9

# Cuts in a graph

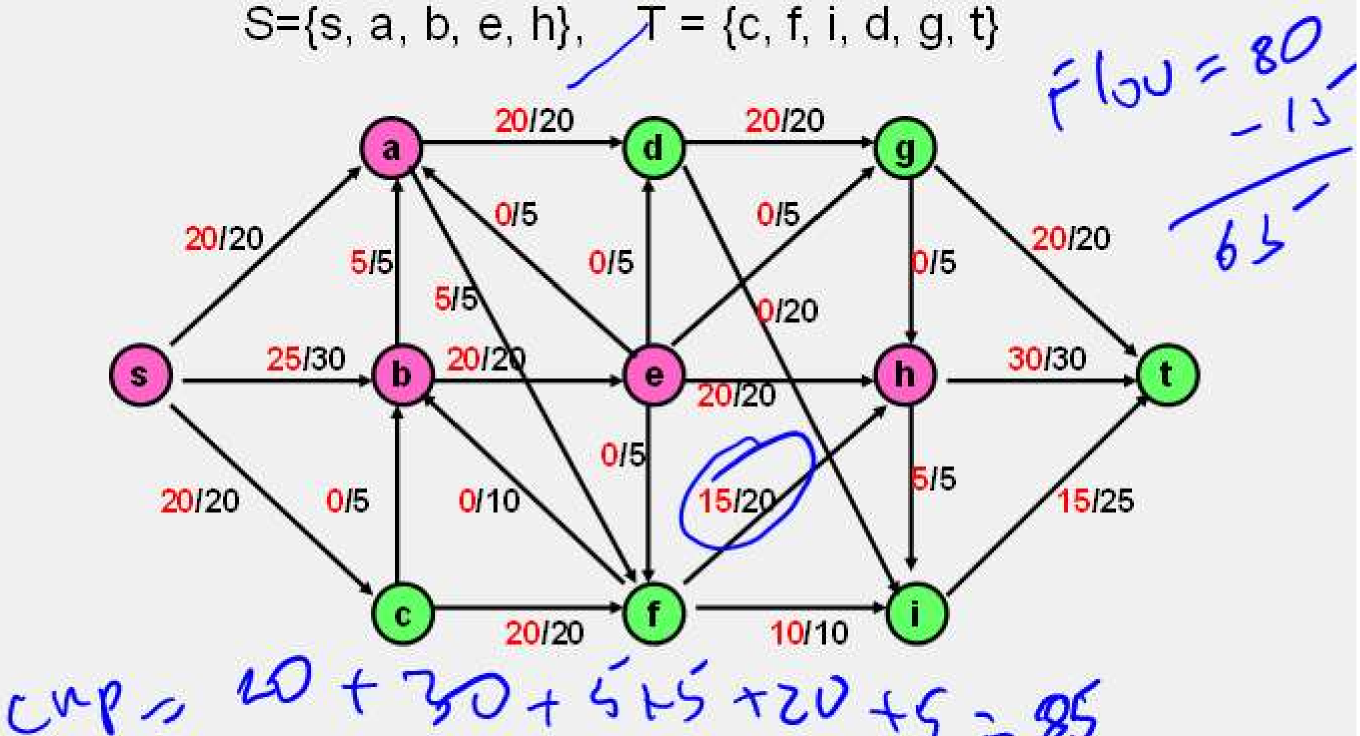
- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S

- Flow(S,T) ≤ Cap(S,T)

10

# What is Cap(S,T) and Flow(S,T)

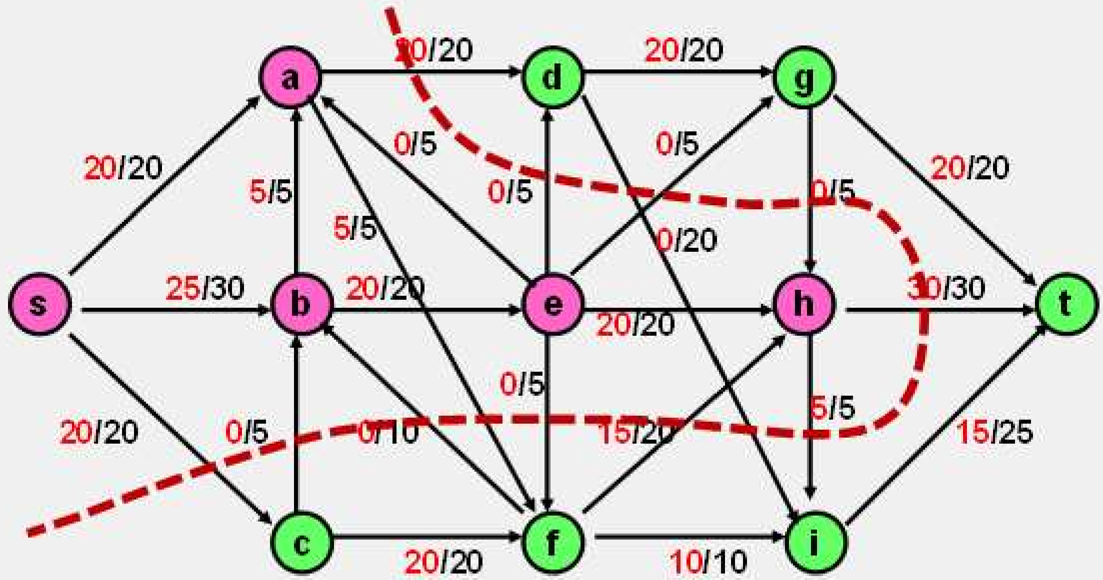
S={s, a, b, e, h}, T = {c, f, i, d, g, t}




11

# What is Cap(S,T) and Flow(S,T)

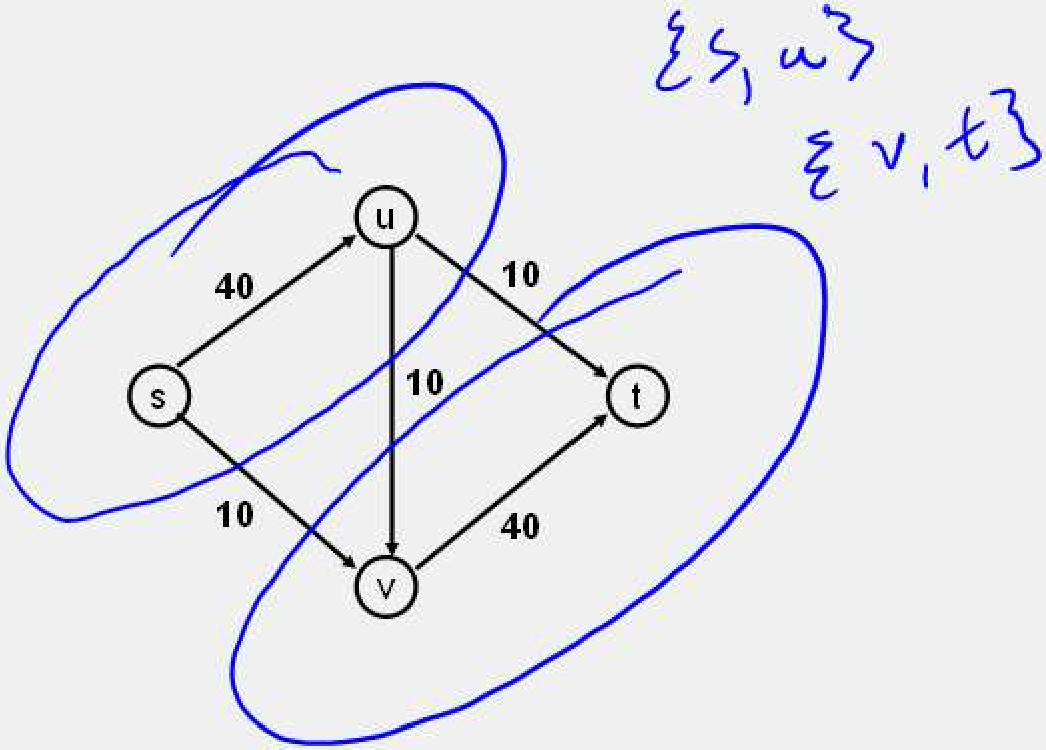
S={s, a, b, e, h}, T = {c, f, i, d, g, t}




Cap(S,T) = 95, Flow(S,T) = 80 – 15 = 65

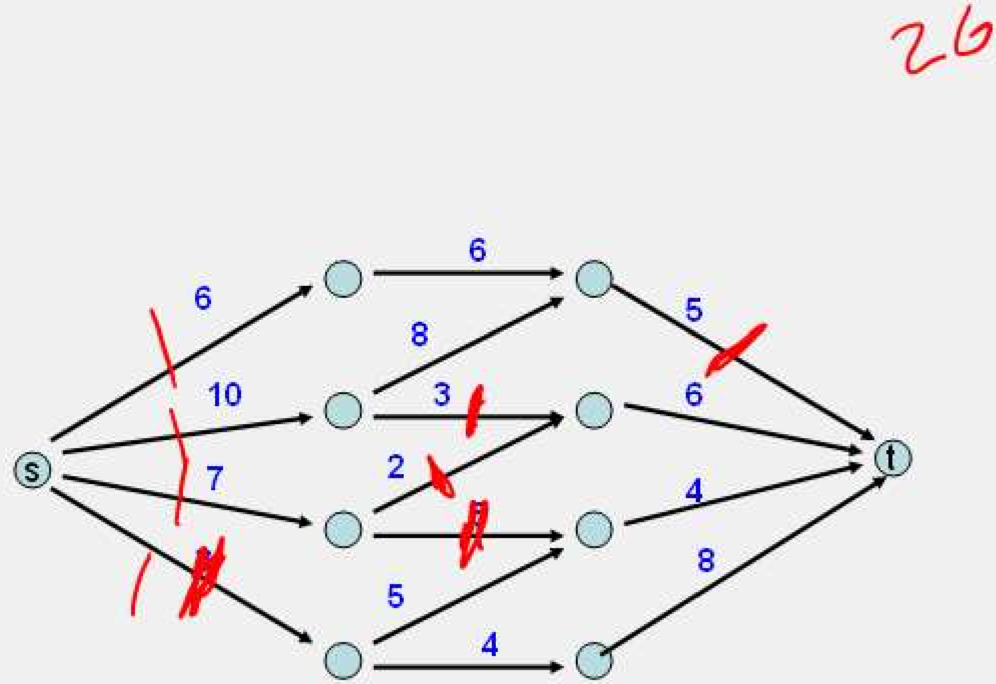
12

# Minimum value cut



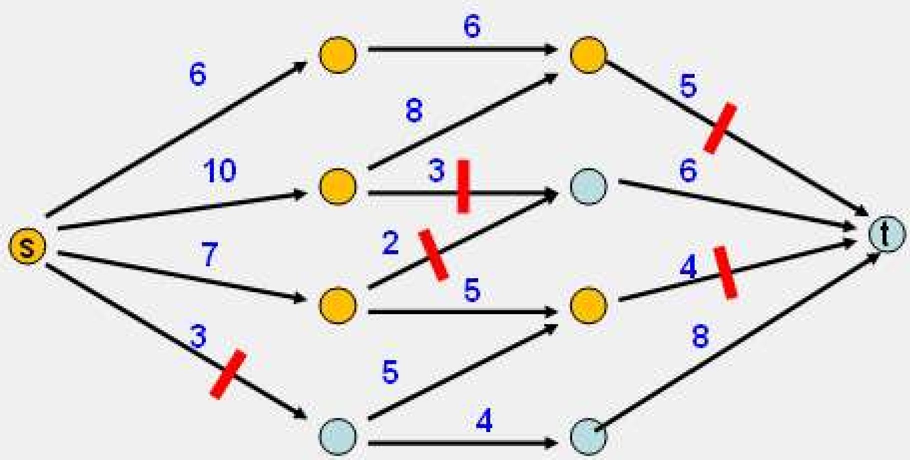

13

# Find a minimum value cut

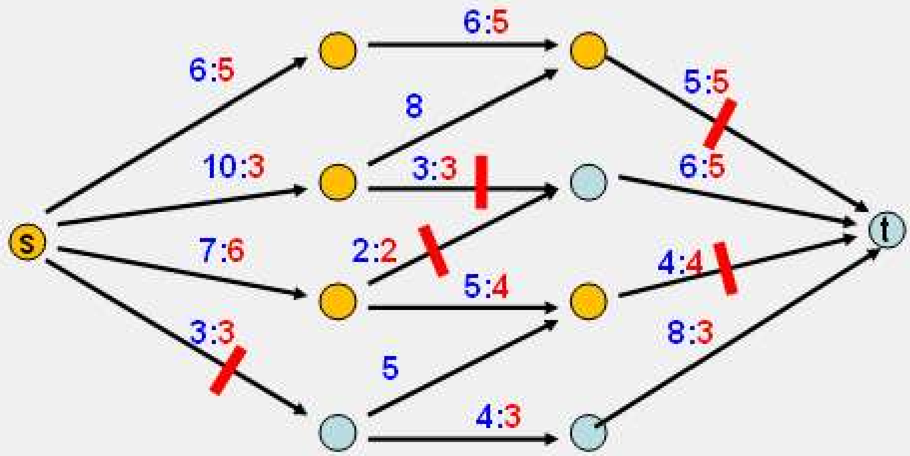



14

# Find a minimum value cut



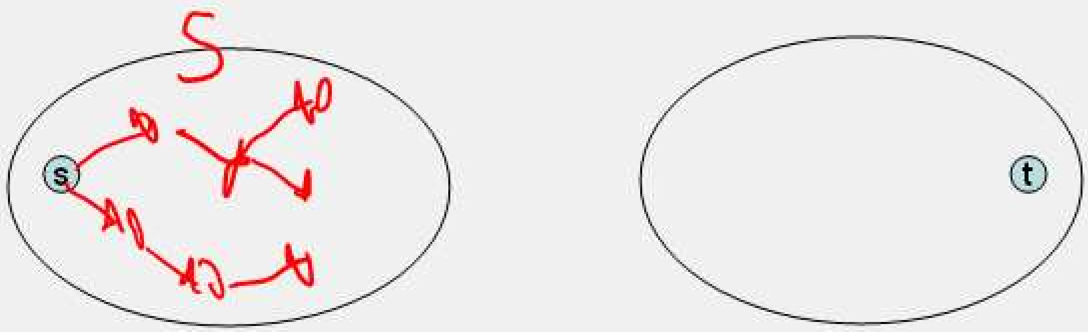
# Find a minimum value cut




16

# MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in $G_R$ reachable from s with paths of positive capacity

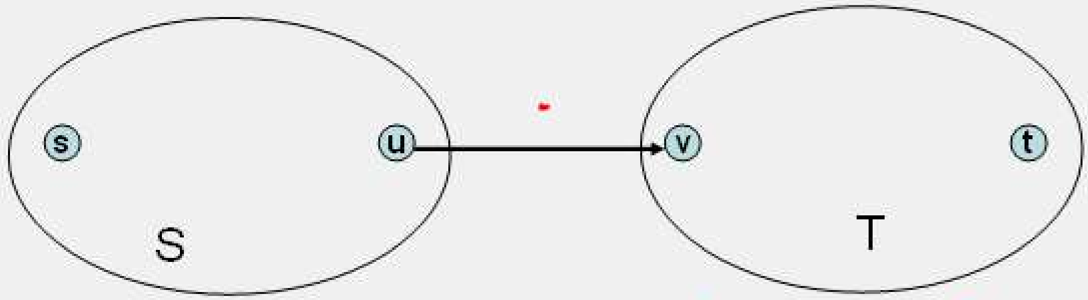



17

Flow(S, T) = Cap(S, T)

# Let S be the set of vertices in $G_R$ reachable from s with paths of positive capacity

cap(u,v) = 0 $G_R$




flow(u,v) = cap(u,v)

flow(v,u) = 0 in G

What can we say about the flows and capacity between u and v?

18

# Max Flow - Min Cut Theorem

1956

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.

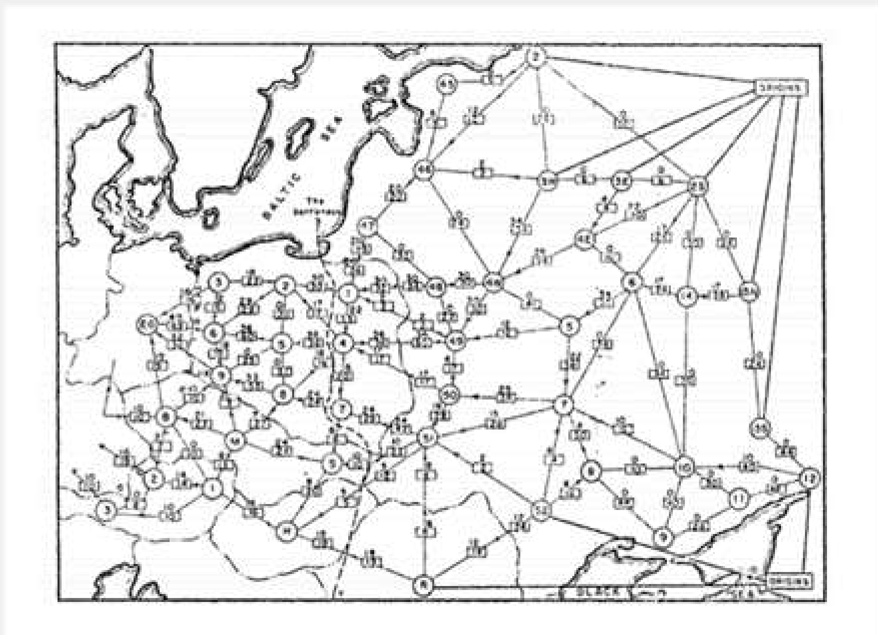
- If we want to find a minimum cut, we begin by looking for a maximum flow.

19

# History

Rand Corp.
1956

- Ford / Fulkerson studied network flow in the context of the Soviet Rail Network



20

# Ford Fulkerson Runtime

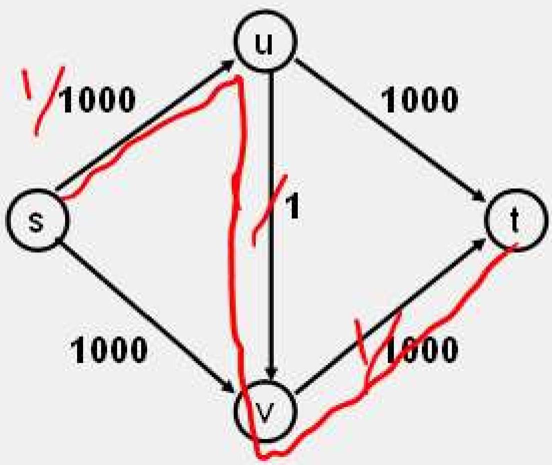
- Cost per phase X number of phases
- Phases
  - Capacity leaving source: C
  - Add at least one unit per phase
- Cost per phase
  - Build residual graph: O(m)
  - Find s-t path in residual: O(m)

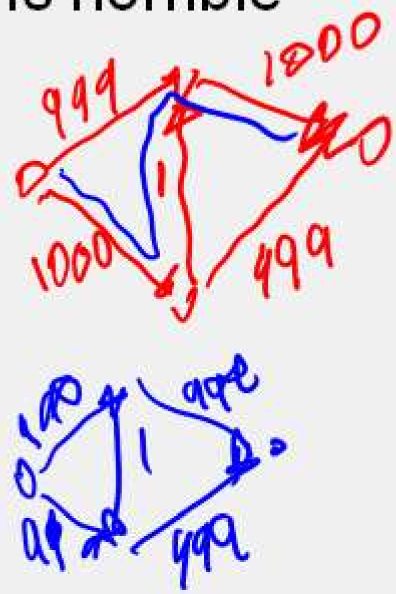
O(Cm)

21

# Performance

- The worst case performance of the Ford-Fulkerson algorithm is horrible



22

# Better methods of finding augmenting paths

- Find the maximum capacity augmenting path
  - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn \log n)$ time algorithm for network flow

23

# Problem Reduction

- ~~Reduce~~ Convert Problem A to Problem B
  - Convert an instance of Problem A to an instance of Problem B
  - Use a solution of Problem B to get a solution to Problem A
- Practical
  - Use a program for Problem B to solve Problem A
- Theoretical
  - Show that Problem B is at least as hard as Problem A

24

# Problem Reduction Examples

- Reduce the problem of finding the Maximum of a set of integers to finding the Minimum of a set of integers

Find the maximum of: -8, -3, -2, -12, -1, -6

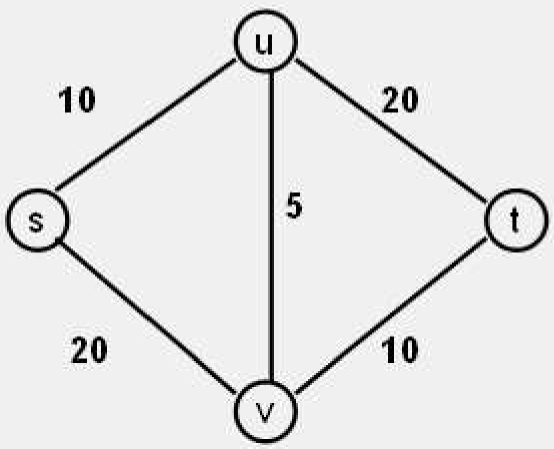
-12

12

Construct an equivalent minimization problem

# Undirected Network Flow

- Undirected graph with edge capacities
- Flow may go either direction along the edges (subject to the capacity constraints)




Construct an equivalent flow problem

26

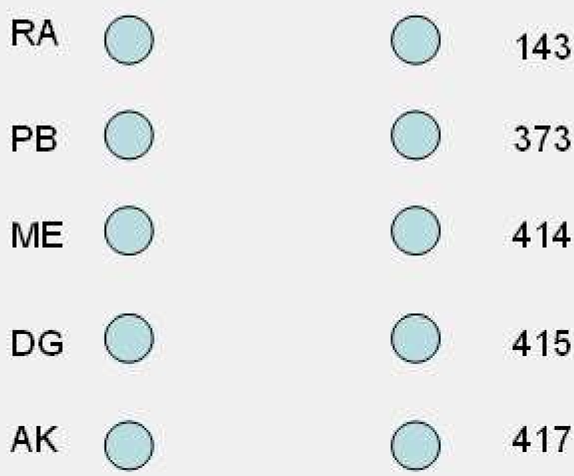
# Bipartite Matching

- A graph G=(V,E) is bipartite if the vertices can be partitioned into disjoints sets X,Y
- A matching M is a subset of the edges that does not share any vertices
- Find a matching as large as possible

27

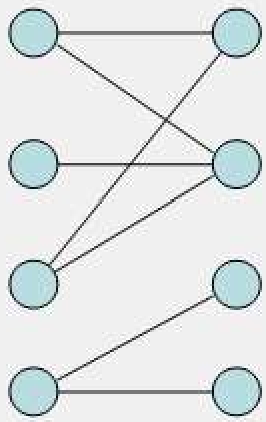
# Application

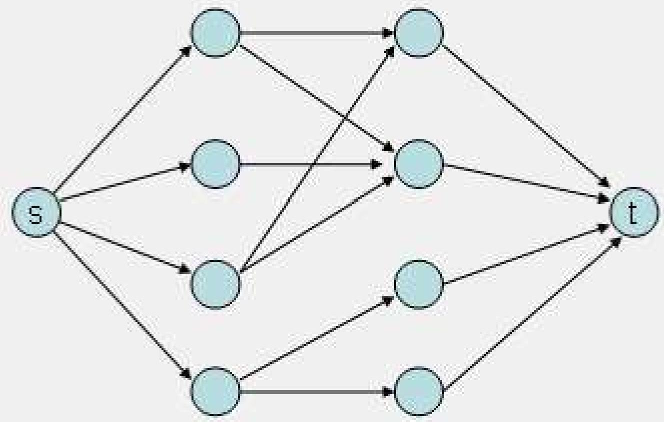
- A collection of teachers
- A collection of courses
- And a graph showing which teachers can teach which courses




28

# Converting Matching to Network Flow






29

# Multi-source network flow

- Multi-source network flow
  - Sources $s_1, s_2, \ldots, s_k$
  - Sinks $t_1, t_2, \ldots, t_j$
- Solve with Single source network flow

30

# Resource Allocation: Assignment of reviewers

- A set of papers $P_1, \ldots, P_n$
- A set of reviewers $R_1, \ldots, R_m$
- Paper $P_i$ requires $A_i$ reviewers
- Reviewer $R_j$ can review $B_j$ papers
- For each reviewer $R_j$, there is a list of paper $L_{j1}, \ldots, L_{jk}$ that $R_j$ is qualified to review

31