

# Lecture24

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# CSE 417

## Algorithms and Complexity

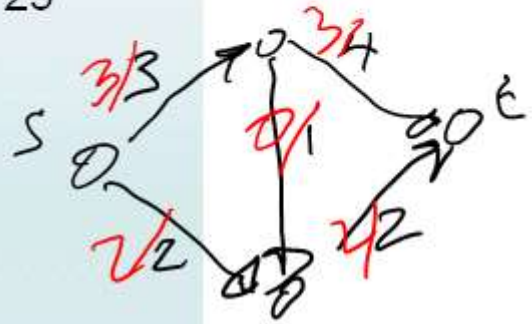
Autumn 2024

Lecture 24

Network Flow, Part 1

# Announcements

- Homework 8, Due Wednesday, Nov 29
- Homework 9, Due Friday, Dec 6
- Final Exam,
  - Monday, December 9, 8:30-10:20 AM



# Network Flow



# Outline

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- ~~Ford Fulkerson Algorithm~~ 1956
- Cuts
- Maxflow-MinCut Theorem
- Applications of NF

# Network Flow Definitions

- Capacity

$$0 \leq f(e) \leq c(e)$$

- Source, Sink



- Capacity Condition

$$0 \leq f(e) \leq c(e)$$

- Conservation Condition

$$\forall v \neq s, t$$

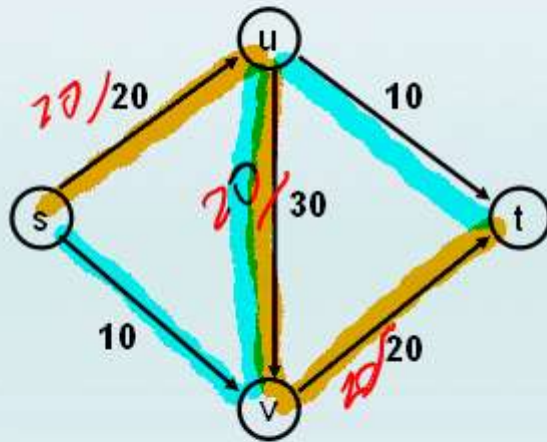
- Value of a flow

$$F = \sum_x f(s, x)$$

$$\sum_w f(v, w) = \sum_t f(v, t)$$



# Flow Example

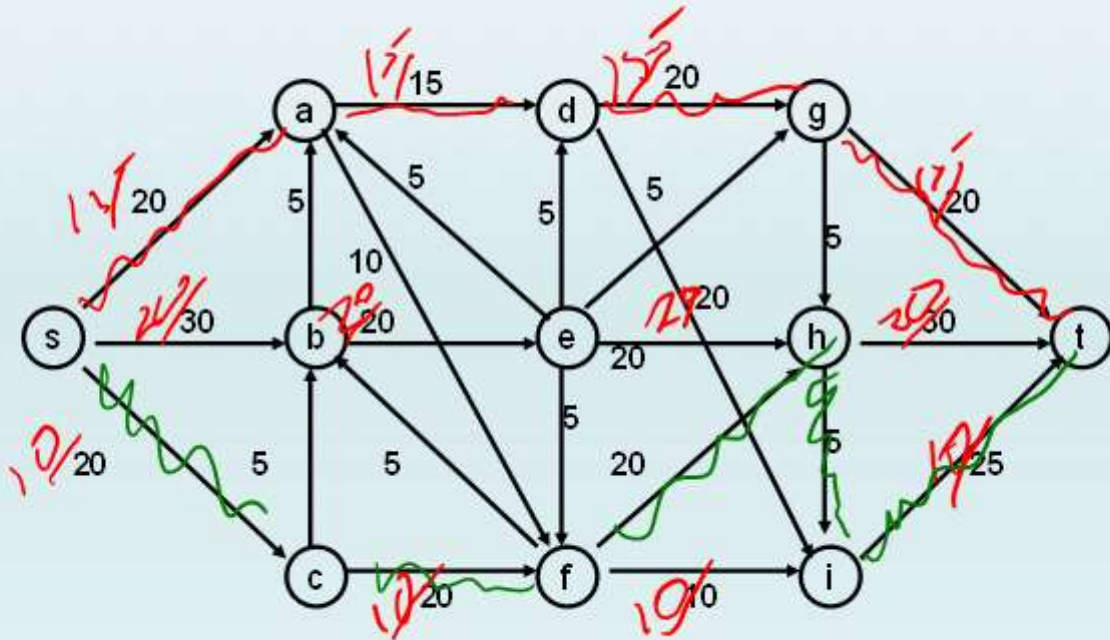




# Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices  $s$  (source) and  $t$  (sink)
- Capacities on the edges,  $c(e) \geq 0$
- Problem, assign flows  $f(e)$  to the edges such that:
  - $0 \leq f(e) \leq c(e)$
  - Flow is conserved at vertices other than  $s$  and  $t$ 
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is as large as possible

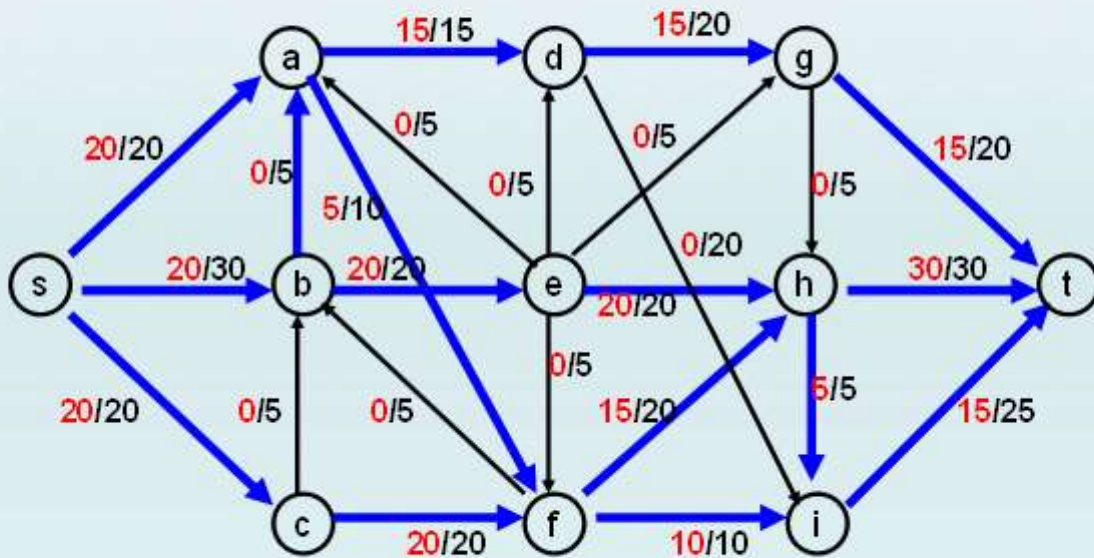
# Find a maximum flow



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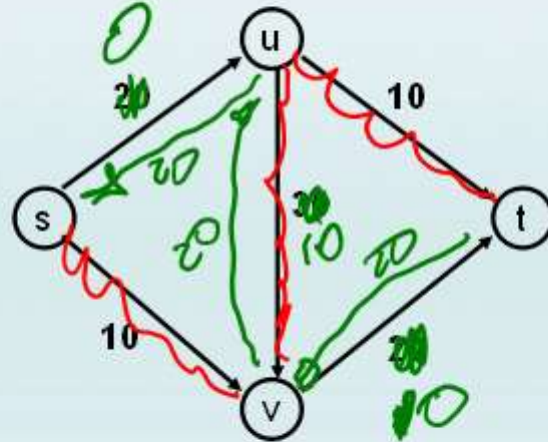
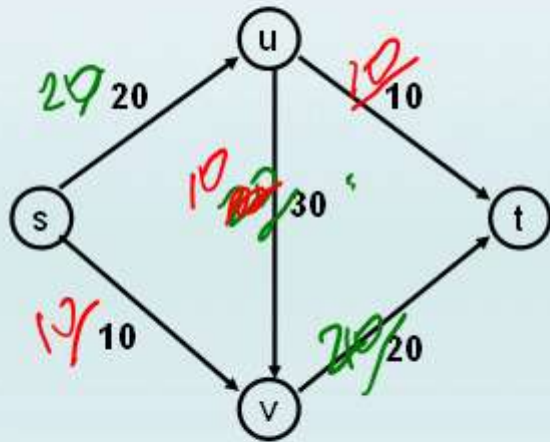
# Find a maximum flow



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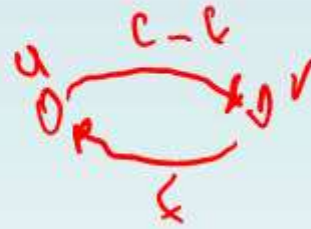
# Flow Example

*Remaining capacity*

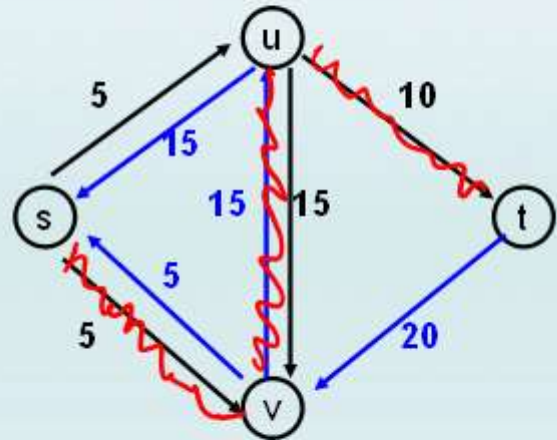
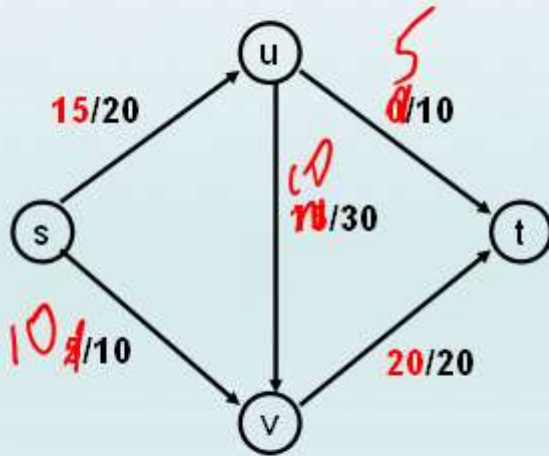


# Residual Graph

- Flow graph showing the remaining capacity
- Flow graph  $G$ , Residual Graph  $G_R$ 
  - $G$ : edge  $e$  from  $u$  to  $v$  with capacity  $c$  and flow  $f$
  - $G_R$ : edge  $e'$  from  $u$  to  $v$  with capacity  $c - f$
  - $G_R$ : edge  $e''$  from  $v$  to  $u$  with capacity  $f$



# Flow assignment and the residual graph



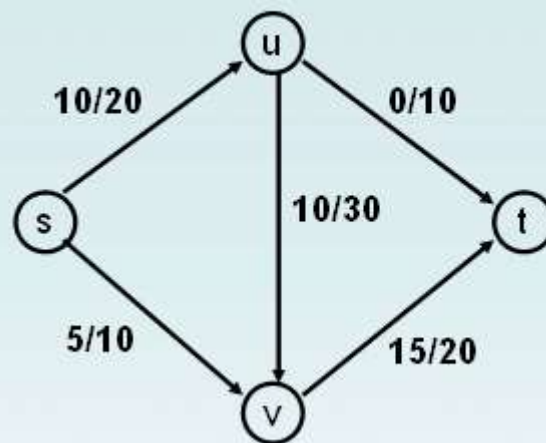
# Augmenting Path Algorithm

- Augmenting path

- Vertices  $v_1, v_2, \dots, v_k$

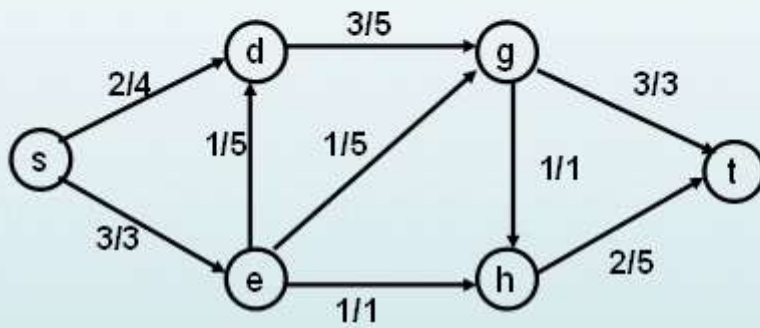
- $v_1 = s, v_k = t$

- Possible to add  $b$  units of flow between  $v_j$  and  $v_{j+1}$  for  $j = 1 \dots k-1$

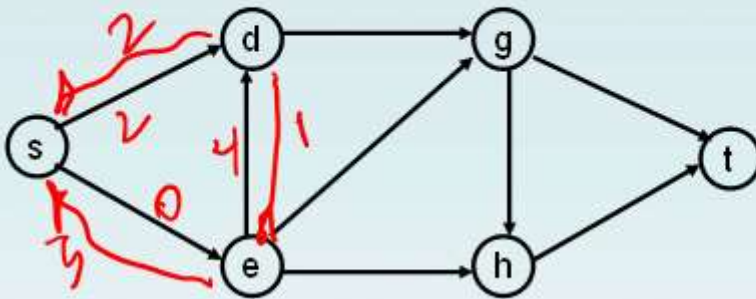


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# Build the residual graph

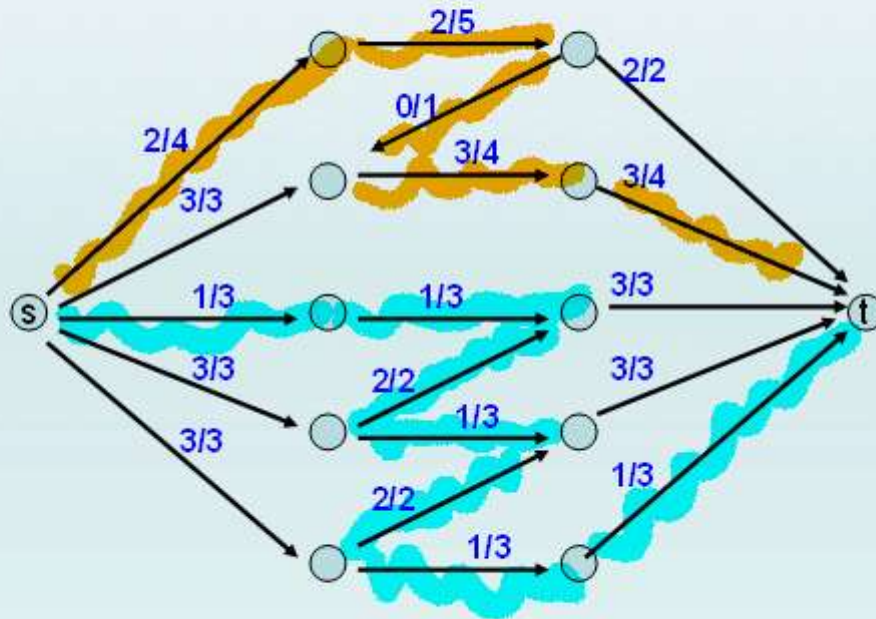


Residual graph:



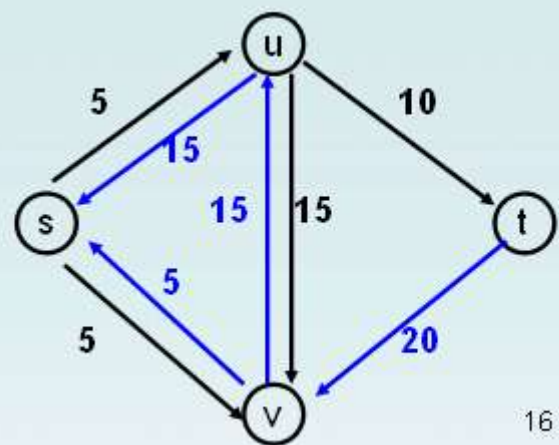
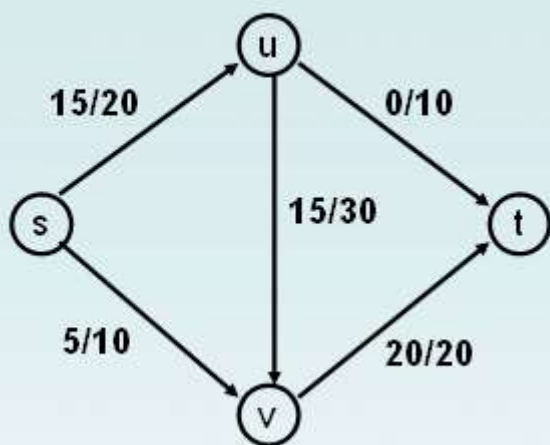


# Find two augmenting paths



# Augmenting Path Lemma

- Let  $P = v_1, v_2, \dots, v_k$  be a path from  $s$  to  $t$  with minimum capacity  $b$  in the residual graph.
- $b$  units of flow can be added along the path  $P$  in the flow graph.



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# Proof

- Add  $b$  units of flow along the path  $P$
- What do we need to verify to show we have a valid flow after we do this?
  - capacity constraint
  - conservation

# Ford-Fulkerson Algorithm (1956)

while not done

    Construct residual graph  $G_R$

    Find an s-t path  $P$  in  $G_R$  with capacity  $b > 0$

    Add  $b$  units along in  $G$

If the sum of the capacities of edges leaving  $s$  is at most  $C$ , then the algorithm takes at most  $C$  iterations