Lecture23

CSE 417: Algorithms with Complexity

Lecture 23 – Autumn 2024

Shortest Paths Problem and Dynamic Programming

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Announcements

- Lecture plans
  - Today: Shortest Paths
  - Friday-Wednesday: Network Flow
  - After Thanksgiving: NP Completeness
- HW 8 Available

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Single Source

Shortest Path Problem

- Dijkstra's Single Source Shortest Paths Algorithm
  - O(mlog n) time, positive cost edges
- Bellman-Ford Algorithm
  - O(mn) time for graphs which can have negative cost edges

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Dynamic Programming

- Express problem as an optimization

- Order subproblems so that results are computed in proper order

Shortest Paths as DP

- $Dist_s[s] = 0$
- $Dist_s[v] = min_w [Dist_s[w] + c_{wv}]$

- How do we order the computation

- Directed Acyclic graph: Topological Sort
- Dijkstra's algorithm determines an order

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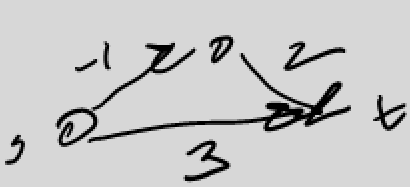
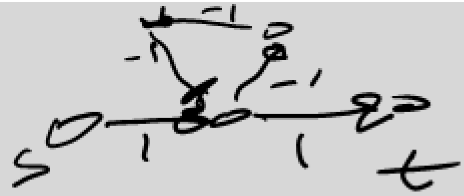


Lemma



- If a graph has no negative cost cycles, then the shortest paths are simple paths

- Shortest paths have at most n-1 edges

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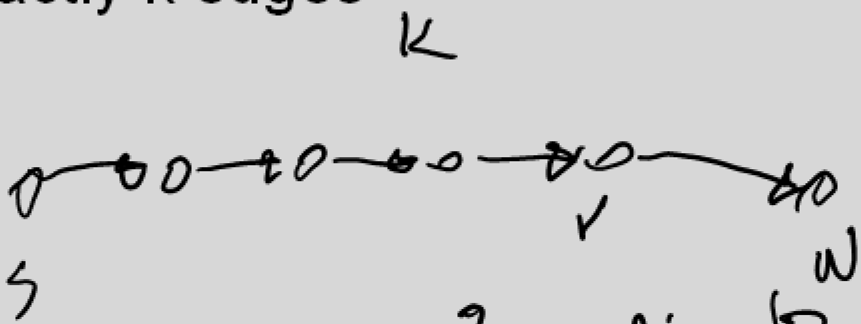
Shortest paths with a given number of edges

- Find the shortest path from s to w with exactly k edges



$$Dist_s[k, w] = \min_v \left(Dist_s[k-1, v] + c_{vw} \right)$$

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Express as a recurrence

- Compute distance from starting vertex s
- $Opt_k(w) = min_x [Opt_{k-1}(x) + c_{xw}]$
- $Opt_0(w)$ = 0 if w = s and infinity otherwise

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Algorithm, Version 1

```
for each w
    M[0, w] = infinity;
M[0, s] = 0;
for i = 1 to n-1
    for each w
        M[i, w] = min_x(M[i-1,x] + cost[x,w]);
```

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Algorithm, Version 2

for each w

    M[0, w] = infinity;

M[0, s] = 0;

for i = 1 to n-1

    for each w

        M[i, w] = min(M[i-1, w], $\min_x$(M[i-1,x] + cost[x,w]));

Bellman-Ford

Algorithm, Version 3

```
for each w
    M[w] = infinity;
M[s] = 0;
for i = 1 to n-1
    for each w
        M[w] = min(M[w], min_x(M[x] + cost[x,w]));
```

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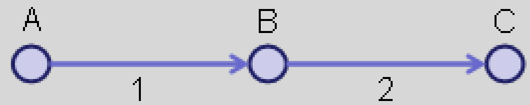
Example:



Alg1

A	B	C
0	∞	∞
∞	1	∞
∞	∞	3

Alg2

A	B	C
0	∞	∞
0	1	∞
0	1	3

Alg3 (B-F)

A	B	C
0	∞	∞
0	1	2

B,C

A	B	C
0	∞	∞
0	1	∞
0	1	3

C,B

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Correctness Proof for Algorithm 3

- Key lemmas, for all w:
  - There exists a path of length M[w] from s to w
  - At the end of iteration i, M[w] ≤ M[i, w]; - Alg 3

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Algorithm, Version 4

```
for each w
    M[w] = infinity;
M[s] = 0;
for i = 1 to n-1
    for each w
        for each x
            if (M[w] > M[x] + cost[x,w])
                P[w] = x;
                M[w] = M[x] + cost[x,w] ;
```

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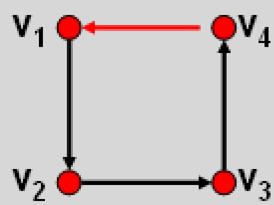
Theorem

If the pointer graph has a cycle, then the graph has a negative cost cycle



Proof: See text.

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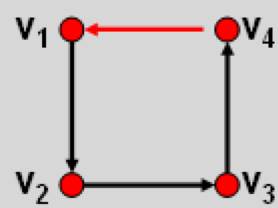
If the pointer graph has a cycle, then the graph has a negative cost cycle

- If P[w] = x then M[w] ≥ M[x] + cost(x,w)
  - Equal when w is updated
  - M[x] could be reduced after update
- Let $v_1, v_2, \dots v_k$ be a cycle in the pointer graph with (v_k, v_1) the last edge added
  - Just before the update
    - $M[v_j] \geq M[v_{j+1}] + cost(v_{j+1}, v_j)$ for $j < k$
    - $M[v_k] > M[v_1] + cost(v_1, v_k)$
  - Adding everything up
    - $0 > cost(v_1,v_2) + cost(v_2,v_3) + \dots + cost(v_k, v_1)$



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Negative Cycles

- If the pointer graph has a cycle, then the graph has a negative cycle
- Therefore: if the graph has no negative cycles, then the pointer graph has no negative cycles

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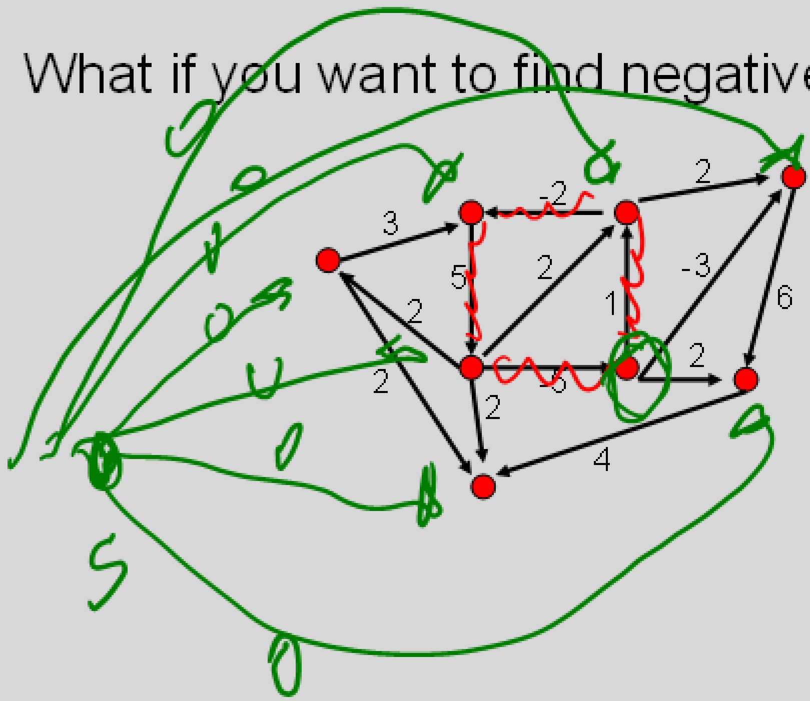
Finding negative cost cycles

- What if you want to find negative cost cycles?



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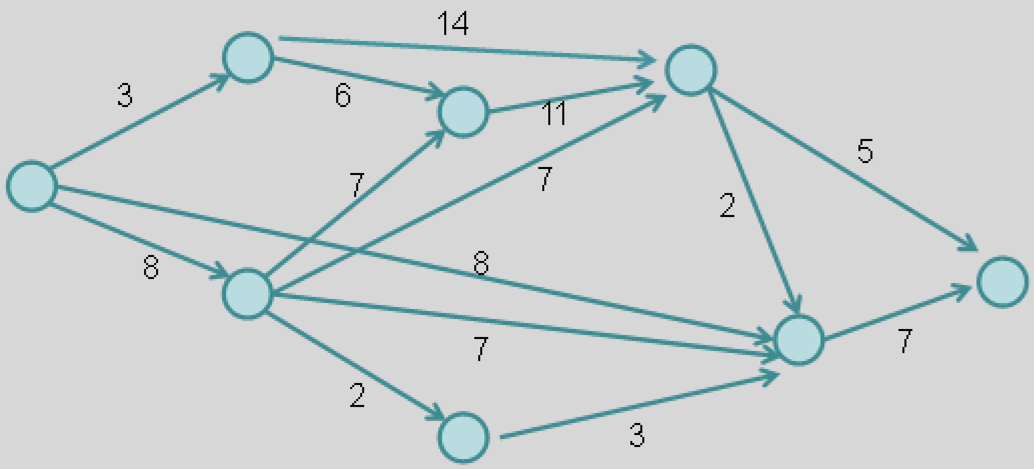
Finding the longest Path in a DAG



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What about finding Longest Paths in a directed graph

- Can we just change Min to Max?

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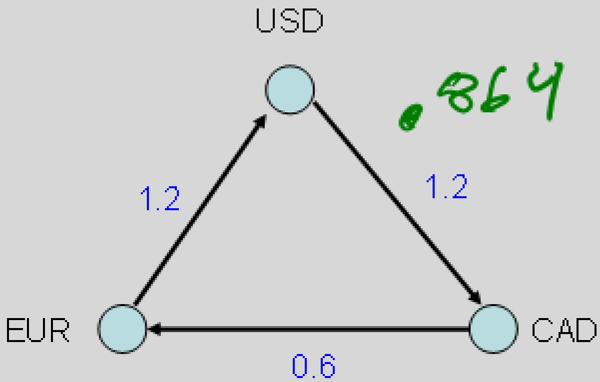
Foreign Exchange Arbitrage



	USD	EUR	CAD
USD	------	0.8	1.2
EUR	1.2	------	1.6
CAD	0.8	0.6	-----