CSE 417: Algorithms with Complexity

Lecture 23 – Autumn 2024

Shortest Paths Problem and Dynamic Programming

Announcements

- Lecture plans
  - Today: Shortest Paths
  - Friday-Wednesday: Network Flow
  - After Thanksgiving: NP Completeness
- HW 8 Available

Shortest Path Problem

- Dijkstra's Single Source Shortest Paths Algorithm
  - O(mlog n) time, positive cost edges
- Bellman-Ford Algorithm
  - O(mn) time for graphs which can have negative cost edges

Dynamic Programming

- Express problem as an optimization

- Order subproblems so that results are computed in proper order

Shortest Paths as DP

- $Dist_s[s] = 0$
- $Dist_s[v] = \min_w [Dist_s[w] + c_{wv}]$

- How do we order the computation

- Directed Acyclic graph: Topological Sort
- Dijkstra's algorithm determines an order

Lemma

- If a graph has no negative cost cycles, then the shortest paths are simple paths

- Shortest paths have at most n-1 edges

Shortest paths with a given number of edges

- Find the shortest path from s to w with exactly k edges

Express as a recurrence

- Compute distance from starting vertex s

- $Opt_k(w) = \min_x [Opt_{k-1}(x) + c_{xw}]$
- $Opt_0(w) = 0$ if $w = s$ and infinity otherwise

Algorithm, Version 1

```
for each w
    M[0, w] = infinity;
M[0, s] = 0;
for i = 1 to n-1
    for each w
        M[i, w] = min_x(M[i-1,x] + cost[x,w]);
```

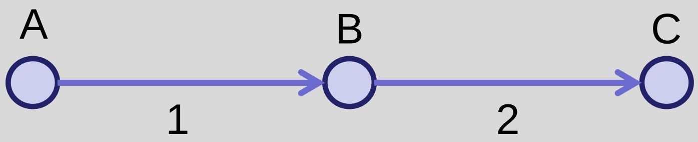
Algorithm, Version 2

```
for each w
    M[0, w] = infinity;
M[0, s] = 0;
for i = 1 to n-1
    for each w
        M[i, w] = min(M[i-1, w], min_x(M[i-1,x] + cost[x,w]));
```

Algorithm, Version 3

```
for each w
    M[w] = infinity;
M[s] = 0;
for i = 1 to n-1
    for each w
        M[w] = min(M[w], min_x(M[x] + cost[x,w]));
```

Example:



A	B	C

A	B	C

A	B	C

A	B	C

Correctness Proof for Algorithm 3

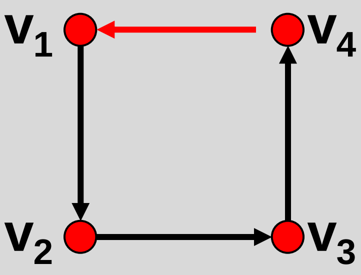
- Key lemmas, for all w:
  - There exists a path of length M[w] from s to w
  - At the end of iteration i, $M[w] \leq M[i, w]$;

Algorithm, Version 4

```
for each w
    M[w] = infinity;
M[s] = 0;
for i = 1 to n-1
    for each w
        for each x
            if (M[w] > M[x] + cost[x,w])
                P[w] = x;
                M[w] = M[x] + cost[x,w] ;
```

Theorem

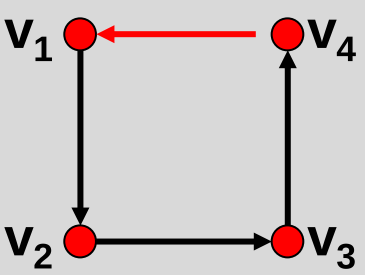
If the pointer graph has a cycle, then the graph has a negative cost cycle



Proof: See text.

If the pointer graph has a cycle, then the graph has a negative cost cycle

- If P[w] = x then M[w] ≥ M[x] + cost(x,w)
  - Equal when w is updated
  - M[x] could be reduced after update
- Let v_1, v_2,…v_k be a cycle in the pointer graph with (v_k,v_1) the last edge added
  - Just before the update
    - $M[v_j] \geq M[v_{j+1}] + cost(v_{j+1}, v_j)$ for $j < k$
    - $M[v_k] > M[v_1] + cost(v_1, v_k)$
  - Adding everything up
    - $0 > cost(v_1,v_2) + cost(v_2,v_3) + \ldots + cost(v_k, v_1)$

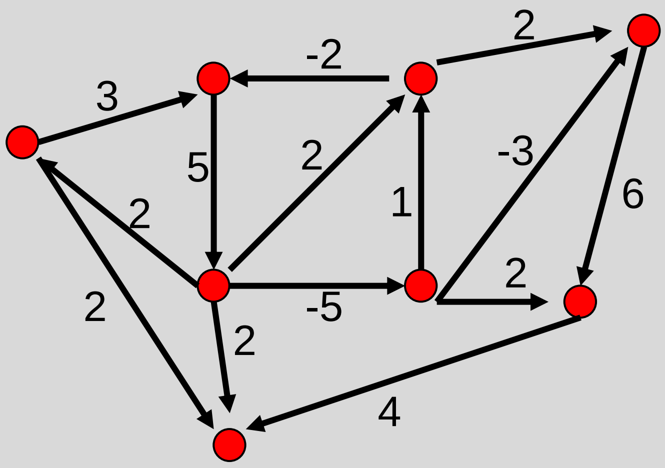


Negative Cycles

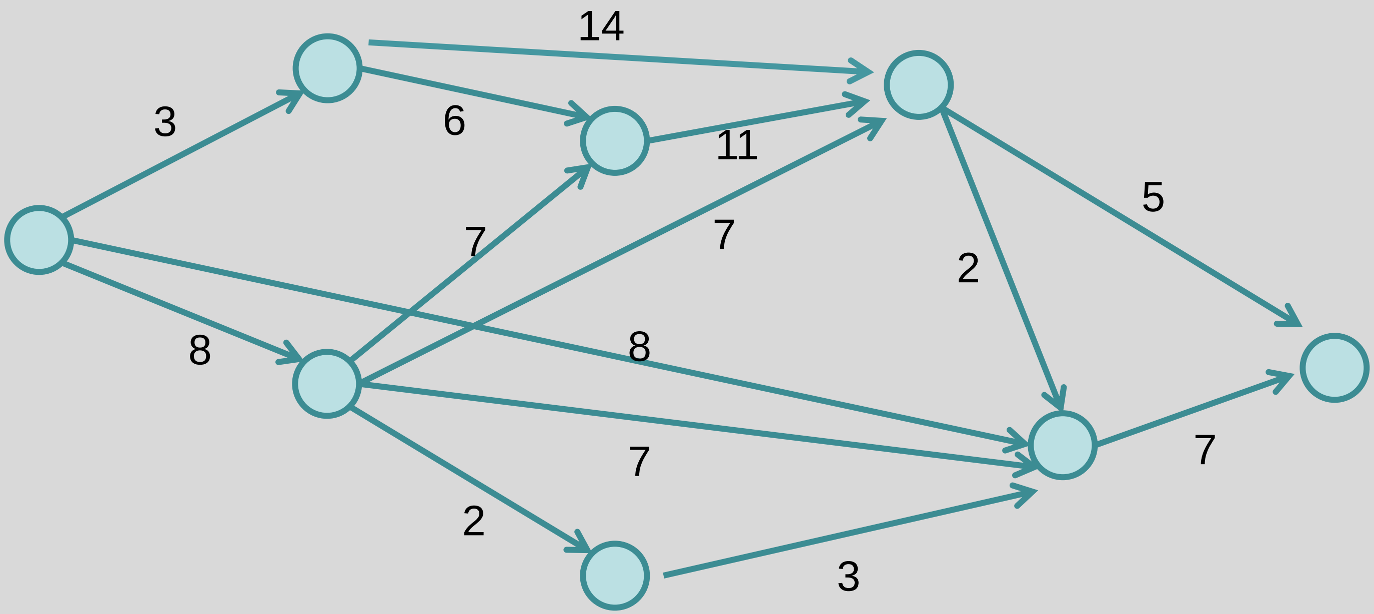
- If the pointer graph has a cycle, then the graph has a negative cycle
- Therefore: if the graph has no negative cycles, then the pointer graph has no negative cycles

Finding negative cost cycles

- What if you want to find negative cost cycles?



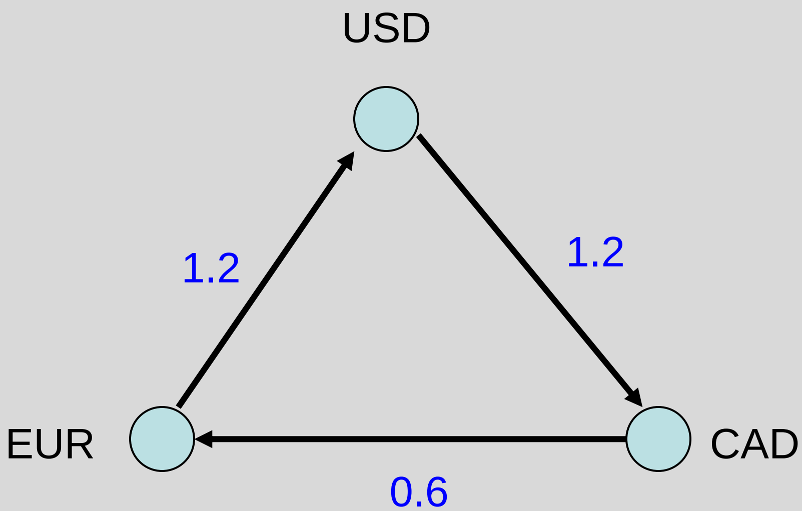
Finding the longest Path in a DAG

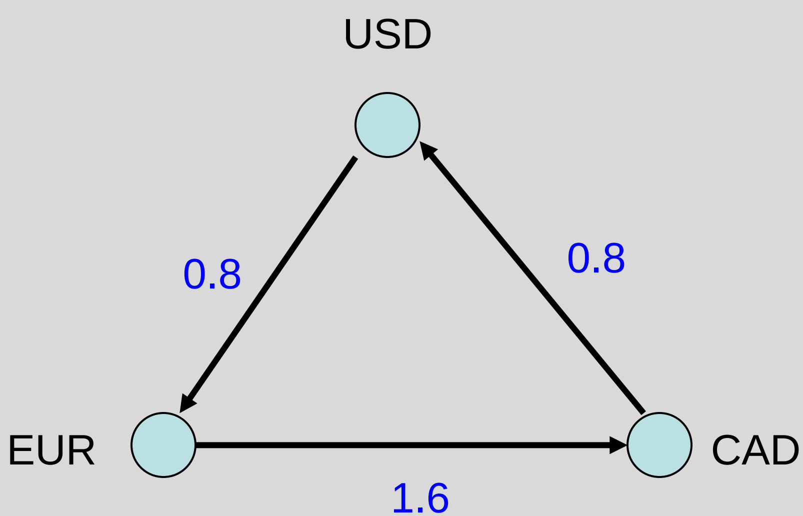


What about finding Longest Paths in a directed graph

- Can we just change Min to Max?

Foreign Exchange Arbitrage





	USD	EUR	CAD
USD	------	0.8	1.2
EUR	1.2	------	1.6
CAD	0.8	0.6	-----