CSE 417
Algorithms and Complexity

Autumn 2024
Lecture 18
Divide and Conquer

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Announcements

- No class on Monday

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Divide and Conquer

- D&C Algorithms
  - MergeSort and QuickSort
  - $O(n^{2.80})$ Matrix Multiplication (Strassen)
  - Integer Multiplication
  - O(n) Median Algorithm
- Today's Algorithms – combining solutions
  - Counting Inversions
  - Closest Pair (in 2D)

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Recurrences

- $T(n) = 2\,T(n/2) + n^2$
- $T(n) = 2\,T(n/2) + n$

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Inversion Problem

- Let $a_1, \ldots, a_n$ be a permutation of 1 . . n
- (a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

- Problem: given a permutation, count the number of inversions
- This can be done easily in $O(n^2)$ time
  - Can we do better?

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Application

- Counting inversions can be use to measure how close ranked preferences are
  - People rank 20 movies, based on their rankings you cluster people who like that same type of movie

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Counting Inversions

11	12	4	1	7	2	3	15	9	5	16	8	6	13	10	14

Count inversions on lower half

Count inversions on upper half

Count the inversions between the halves

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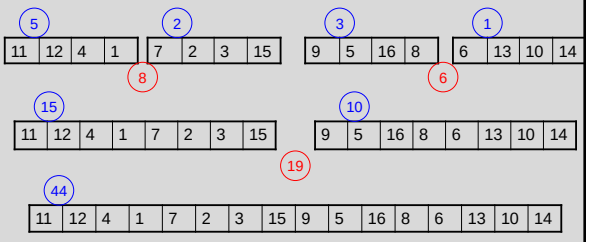
Count the Inversions



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Problem – how do we count inversions between sub problems in O(n) time?

- Solution – Count inversions while merging

1	2	3	4	7	11	12	15

5	6	8	9	10	13	14	16

Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution

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Use the merge algorithm to count inversions

1	4	11	12

2	3	7	15

5	8	9	16

6	10	13	14

Indicate the number of inversions for each element detected when merging

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Inversions

- Counting inversions between two sorted lists
  - O(1) per element to count inversions

x	x	x	x	x	x	x	x

y	y	y	y	y	y	y	y

z	z	z	z	z	z	z	z	z	z	z	z	z	z	z	z

- Algorithm summary
  - Satisfies the "Standard recurrence"
  - T(n) = 2 T(n/2) + cn

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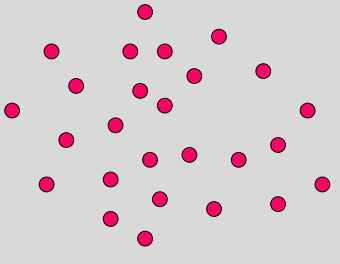
Closest Pair Problem (2D)

- Given a set of points find the pair of points p, q that minimizes dist(p, q)



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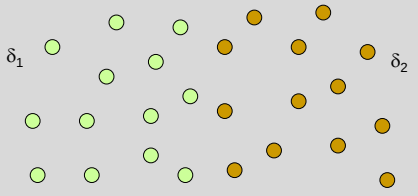
Divide and conquer

- If we solve the problem on two subsets, does it help? (Separate by median x coordinate)



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Packing Lemma

Suppose that the minimum distance between points is at least δ, what is the maximum number of points that can be packed in a ball of radius δ?

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Combining Solutions

- Suppose the minimum separation from the sub problems is δ
- In looking for cross set closest pairs, we only need to consider points with δ of the boundary
- How many cross border interactions do we need to test?

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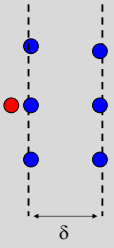
A packing lemma bounds the number of distances to check



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Details

- Preprocessing: sort points by y
- Merge step
  - Select points in boundary zone
  - For each point in the boundary
    - Find highest point on the other side that is at most δ above
    - Find lowest point on the other side that is at most δ below
    - Compare with the points in this interval (there are at most 6)

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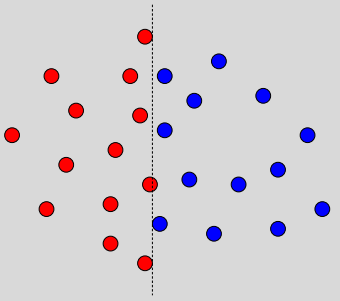
Identify the pairs of points that are compared in the merge step following the recursive calls



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Algorithm run time

- After preprocessing:
  - $T(n) = cn + 2\,T(n/2)$

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Divide and Conquer Summary

- Performance of Divide and Conquer
  - Reduce the number or size of subproblems
  - Reduce the amount of work in combining solutions

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