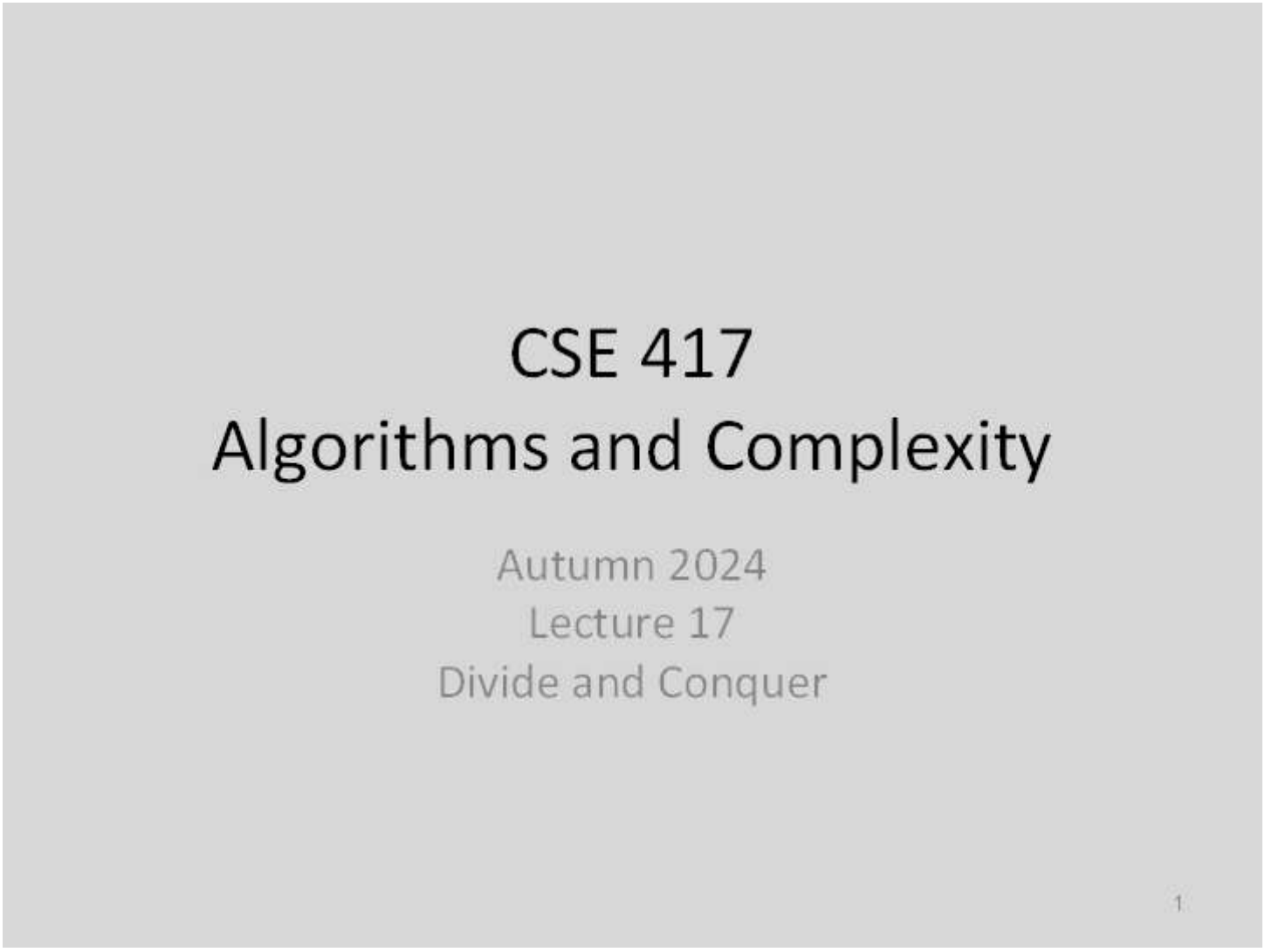
Lecture17


CSE 417
Algorithms and Complexity
Autumn 2024
Lecture 17
Divide and Conquer
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Announcements

- HW 6 available, due Friday
- Schedule
  - Today – sorting and divide and conquer
  - Wednesday – more divide and conquer
  - Friday – dynamic programming

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Divide and Conquer for Sorting

- Merge Sort

```
Array MSort(Array a, int n){
   if (n ≤ 1)  return a;
   return Merge(MSort(a[0 .. n/2], n/2),  MSort(a[n/2+1 .. n-1], n/2);
}
```

- $T(n) = 2T(n/2) + n; T(1) = 1;$

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Quicksort [Tony Hoare, 1959]

QuickSort(S):

1. Pick an element v in **S**. This is the ***pivot*** value.
2. Partition **S**-{v} into two disjoint subsets, $\mathbf{S}_1$ and $\mathbf{S}_2$ such that:
   - elements in $\mathbf{S}_1$ are all $< v$
   - elements in $\mathbf{S}_2$ are all $> v$
3. Return concatenation of QuickSort($\mathbf{S}_1$), v, QuickSort($\mathbf{S}_2$)

Recursion ends if Quicksort( ) receives an array of length 0 or 1.

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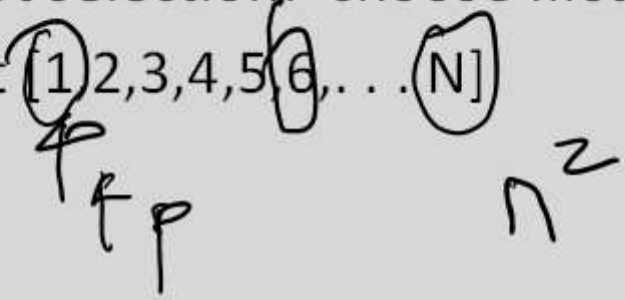
Quicksort – worst case

- Pivot selection: choose first element
- Sort [1,2,3,4,5,6,. . .,N]



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Quicksort - pragmatics

- Pivot selection rules
  - Median of first, middle, and last
  - Choose random element
- In place implementation
- Algorithm engineering for partitioning
- Recursion cutoff for small problems

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Average case analysis for Quicksort

- All inputs equally likely
  - Or random elements used for pivot
  - Or input is randomly shuffled
- QS(n) = average number of comparisons for Quicksort on input of size n.

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Building a recurrence

Pivot chosen at random. The chance of having i elements less than the pivot is 1/n.

$$T(n) = (n-1) + \frac{1}{n}\sum_{i=0}^{n-1}(T(i) + T(n-i-1)).$$

$$T(n) = (n-1) + \frac{2}{n}\sum_{i=1}^{n-1} T(i)$$

Solution: $T(N) \approx 2\, n \ln n$

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Computing the Median

- Given n numbers, find the number of rank n/2
- One approach is sorting
  - Sort the elements, and choose the middle one
  - Can you do better?

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Problem generalization

- *Selection*, given n numbers and an integer k, find the k-th largest

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Select(A, k)

```
Select(A, k){
        Choose element x from A
        S1 = {y in A | y < x}
        S2 = {y in A | y > x}
        S3 = {y in A | y = x}
        if (|S2| >= k)
                return Select(S2, k)
        else if (|S2| + |S3| >= k)
                return x
        else
                return Select(S1, k - |S2| - |S3|)
}
```

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Randomized Selection

- Choose the element at random
- Analysis can show that the algorithm has expected run time O(n)

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$|x|<1 \quad \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

Deterministic Selection

$\frac{1}{1-\frac{3}{4}}$

- What is the run time of select if we can guarantee that choose finds an x such that $|S_1| < 3n/4$ and $|S_2| < 3n/4$ in O(n) time

$T(n) = T(\frac{3}{4}n) + n = T(\frac{1}{16}n) + \frac{3}{4}n + n$

$= \sum_{i=1}^{k} \left(\frac{3}{4}\right)^i \leq n \sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^i =$

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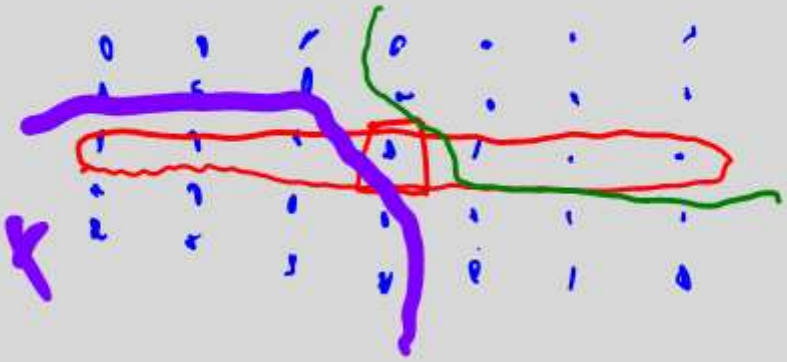
BFPRT Algorithm

- A very clever choose algorithm . . .

Split into n/5 sets of size 5
M be the set of medians of these sets
Let x be the median of M



BFPRT runtime

$|S_1| < 3n/4$, $|S_2| < 3n/4$

Split into n/5 sets of size 5 $O(n)$

M be the set of medians of these sets $O(n)$

x be the median of M $T(\frac{n}{5})$

Construct S_1 and S_2 $O(n)$

Recursive call in S_1 or S_2 $T(\frac{3}{4}n)$

$T(n) \leq T(\frac{3}{4}n) + T(\frac{n}{5}) + cn$

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BFPRT Recurrence

$$T(n) \leq T(3n/4) + T(n/5) + c\,n$$

Suppose $T(k) \leq 20ck$ for $k < n$

$$T(n) \leq 20c\frac{3n}{4} + 20c\frac{n}{5} + cn$$

$$= 15cn + 4cn + cn$$

$$= 20cn$$

Prove that $T(n) \leq 20\,c\,n$

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Inversion Problem

- Let $a_1, \ldots a_n$ be a permutation of 1 . . n
- (a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

- Problem: given a permutation, count the number of inversions
- This can be done easily in $O(n^2)$ time
  - Can we do better?

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Application

- Counting inversions can be use to measure how close ranked preferences are
  - People rank 20 movies, based on their rankings you cluster people who like that same type of movie

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Counting Inversions

11	12	4	1	7	2	3	15	9	5	16	8	6	13	10	14

Count inversions on lower half

Count inversions on upper half

Count the inversions between the halves

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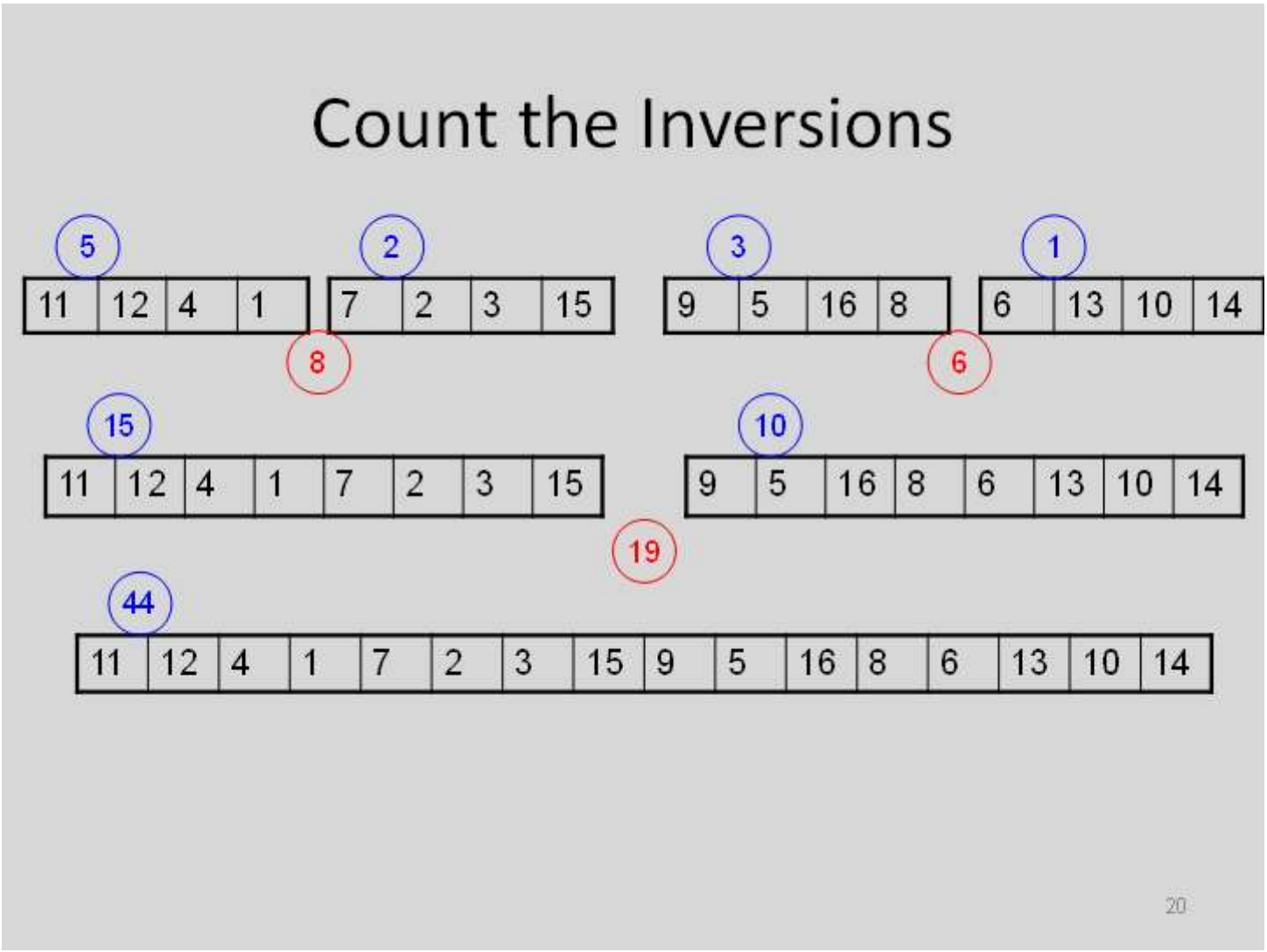
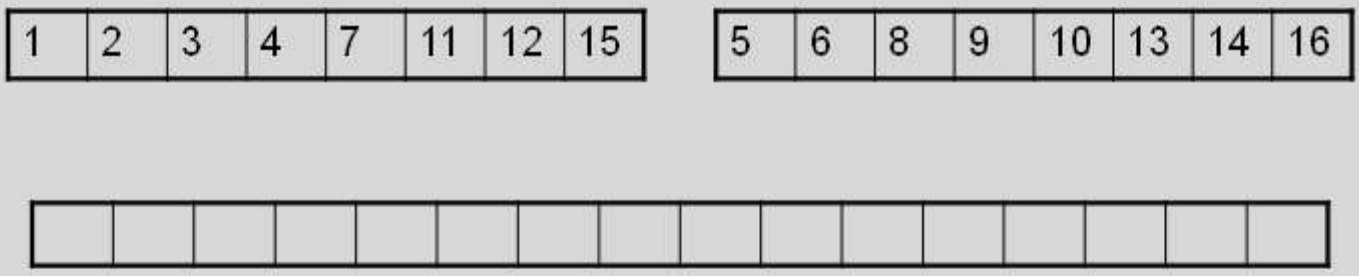

Count the Inversions
5
11 12 4 1
2
7 2 3 15
8
3
9 5 16 8
1
6 13 10 14
6
15
11 12 4 1 7 2 3 15
10
9 5 16 8 6 13 10 14
19
44
11 12 4 1 7 2 3 15 9 5 16 8 6 13 10 14
20

Problem – how do we count inversions between sub problems in O(n) time?

- Solution – Count inversions while merging



Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution

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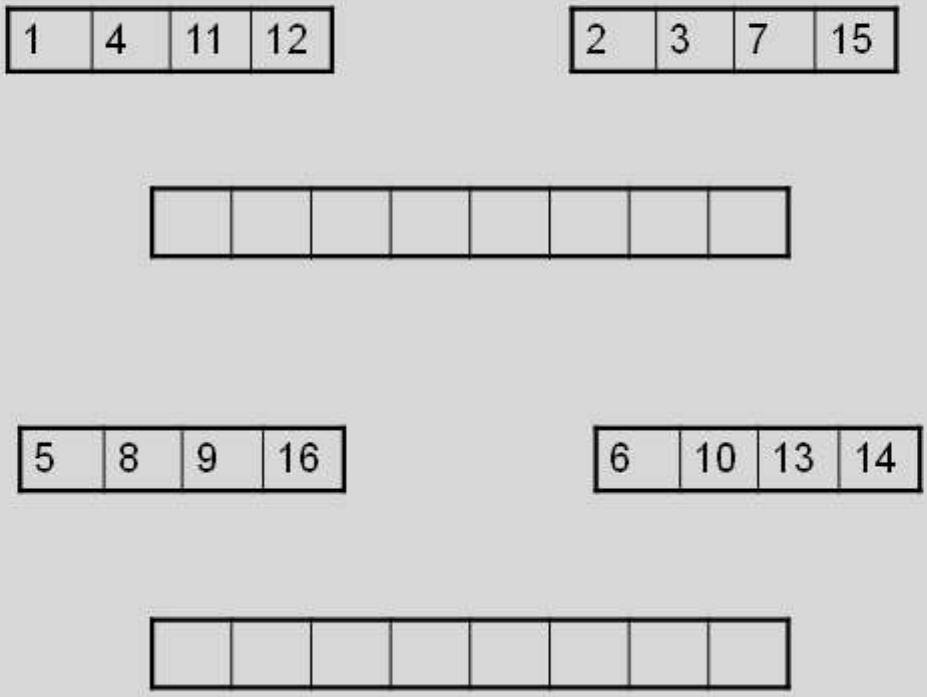
Use the merge algorithm to count inversions



Indicate the number of inversions for each element detected when merging

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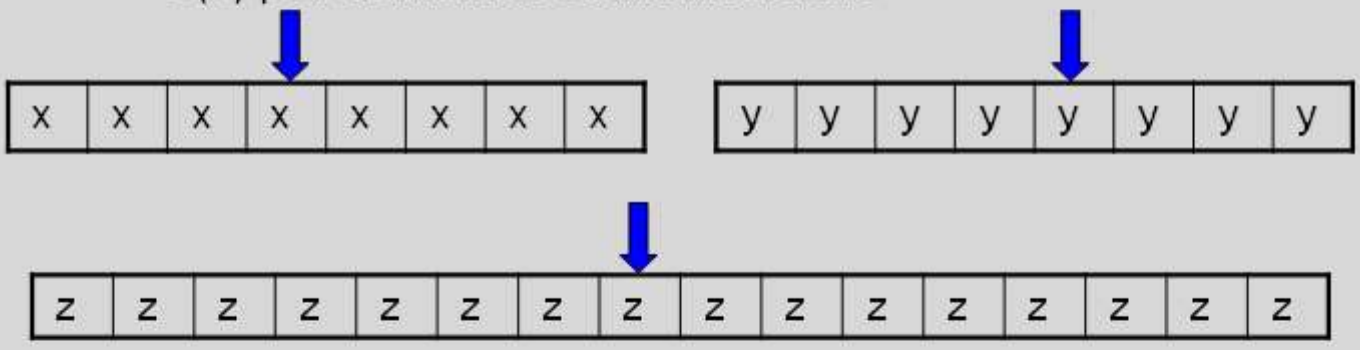
Inversions

- Counting inversions between two sorted lists
  - O(1) per element to count inversions



- Algorithm summary
  - Satisfies the "Standard recurrence"
  - $T(n) = 2\,T(n/2) + cn$

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