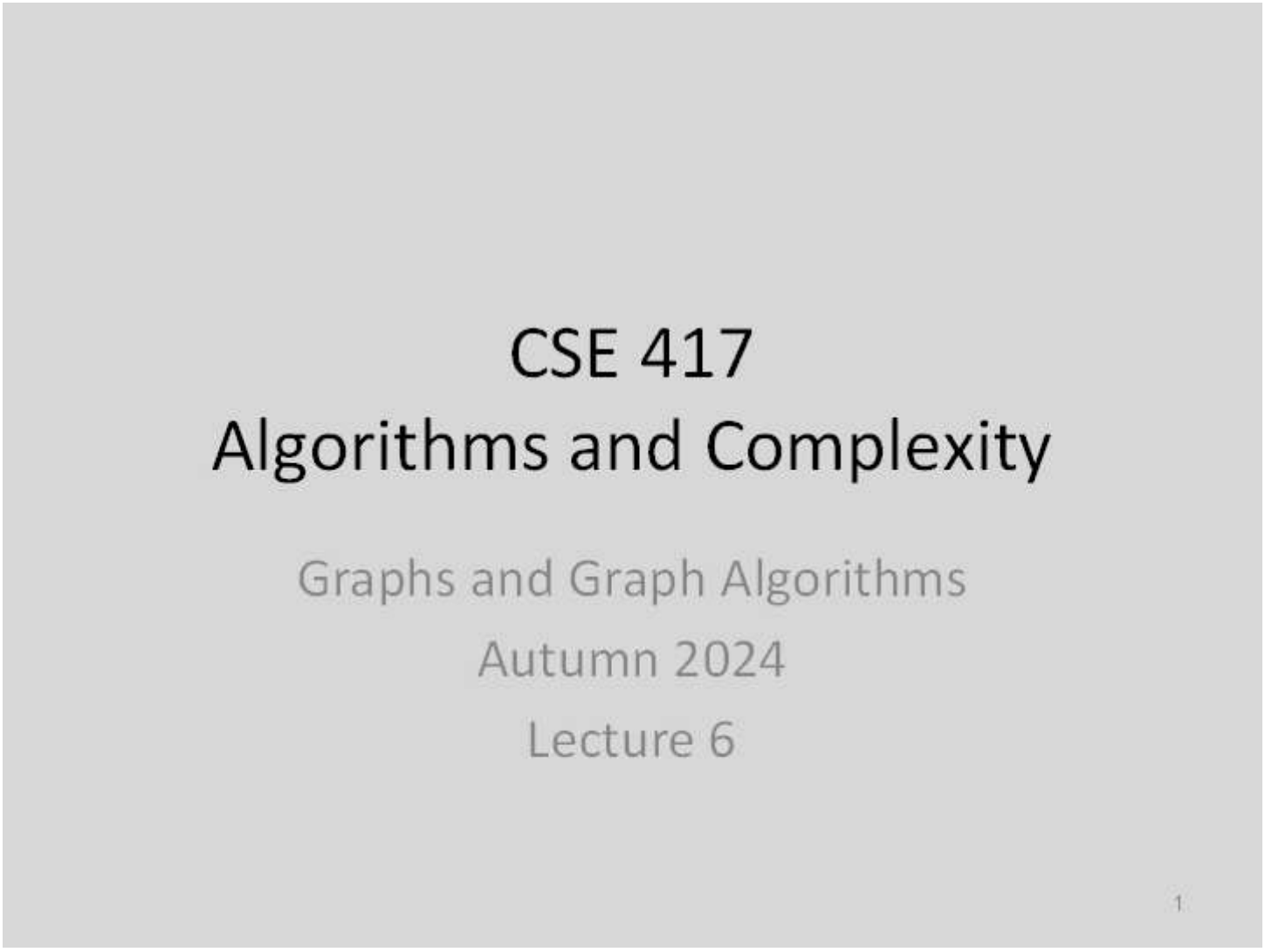
Lecture06


CSE 417
Algorithms and Complexity
Graphs and Graph Algorithms
Autumn 2024
Lecture 6
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Announcements

- Reading
  - Chapter 3
  - Start on Chapter 4
- Homework 2

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Graph Theory

- G = (V, E)
  - V: vertices, $|V| = n$
  - E: edges, $|E| = m$
- Undirected graphs
  - Edges sets of two vertices {u, v}
- Directed graphs
  - Edges ordered pairs (u, v)
- Many other flavors
  - Edge / vertices weights
  - Parallel edges
  - Self loops

- Path: $v_1, v_2, \ldots, v_k$, with (v_i, v_{i+1}) in E
  - **Simple Path**
  - **Cycle**
  - **Simple Cycle**
- Neighborhood
  - N(v)
- Distance
- Connectivity
  - **Undirected**
  - **Directed (strong connectivity)**
- Trees
  - **Rooted**
  - **Unrooted**

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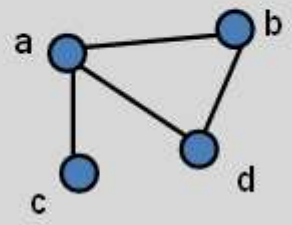
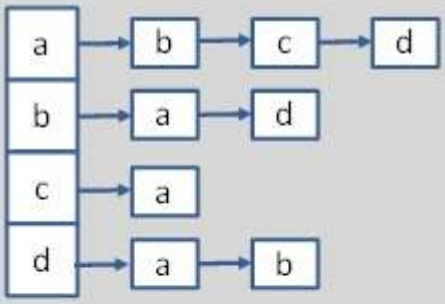
Graph Representation



$V = \{ a, b, c, d \}$

$E = \{ \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\} \}$



Adjacency List

$O(n + m)$ space

	1	1	1
1		0	1
1	0		0
1	1	0	

Incidence Matrix

$O(n^2)$ space

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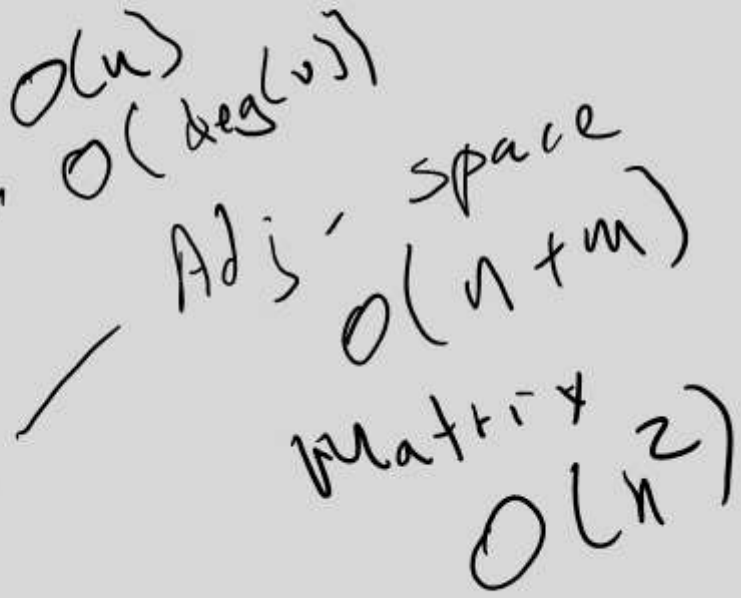
Implementation Issues

- Graph with n vertices, m edges
- Operations
  - Lookup edge
  - Add edge
  - Enumeration edges
  - Initialize graph
- Space requirements



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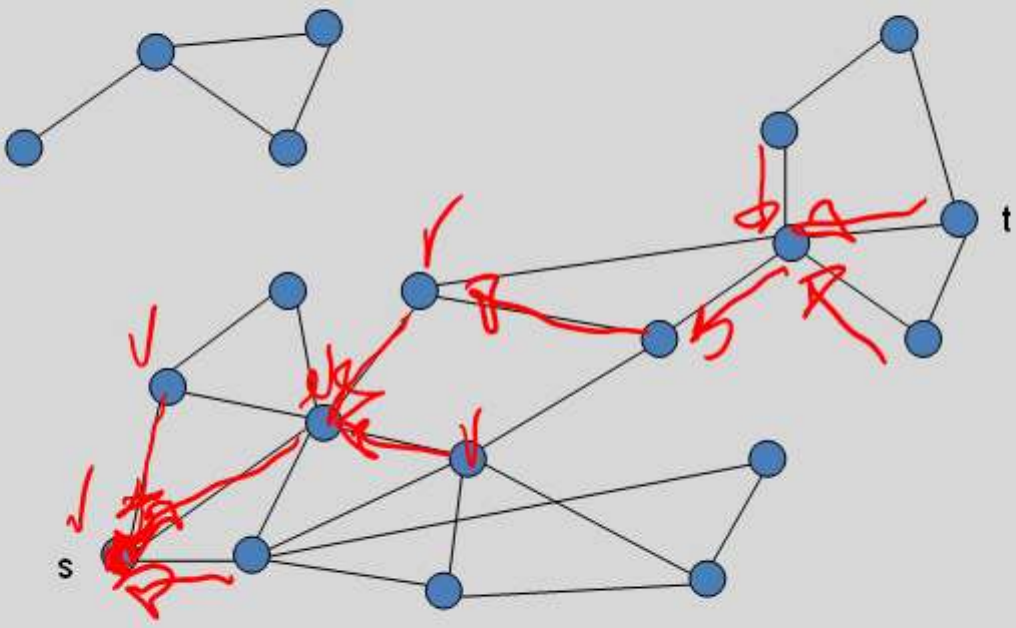
Graph search

- Find a path from s to t

```
S = {s}; mark s
while S is not empty
        u = Select(S)
        if (u == t) then path found
        foreach v in N(u)
                if v is unmarked
                        mark v
                        Add(S, v)
                        Pred[v] = u
```

Graph Search



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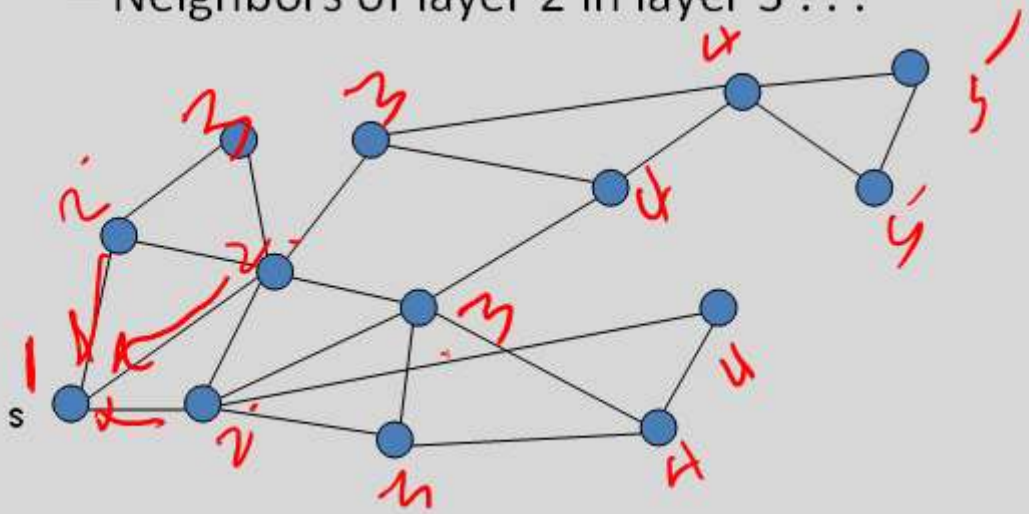
Breadth first search

- Explore vertices in layers
  - s in layer 1
  - Neighbors of s in layer 2
  - Neighbors of layer 2 in layer 3 . . .



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Breadth First Search

- Build a BFS tree from s

```
Initialize Level[v] = -1 for all v;
Q = {s}
Level[s] = 1;
while Q is not empty
        u = Q.Dequeue()
        foreach v in N(u)
                if (Level[v] == -1)
                        Q.Enqueue(v)
                        Pred[v] = u
                        Level[v] = Level[u] + 1
```

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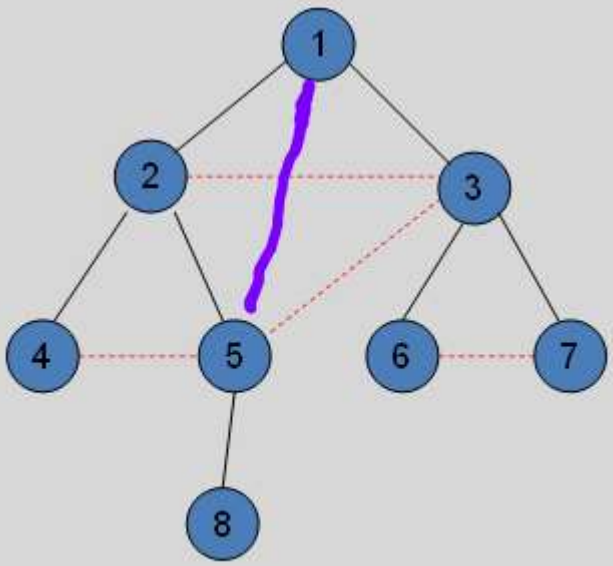
Key observation

- All edges go between vertices on the same layer or adjacent layers



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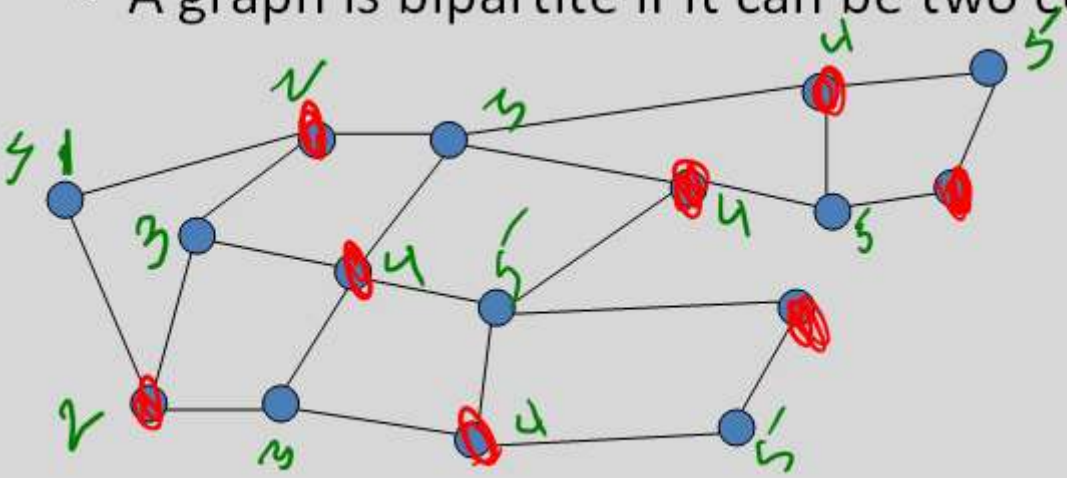
Bipartite Graphs

- A graph V is bipartite if V can be partitioned into V_1, V_2 such that all edges go between V_1 and V_2
- A graph is bipartite if it can be two colored



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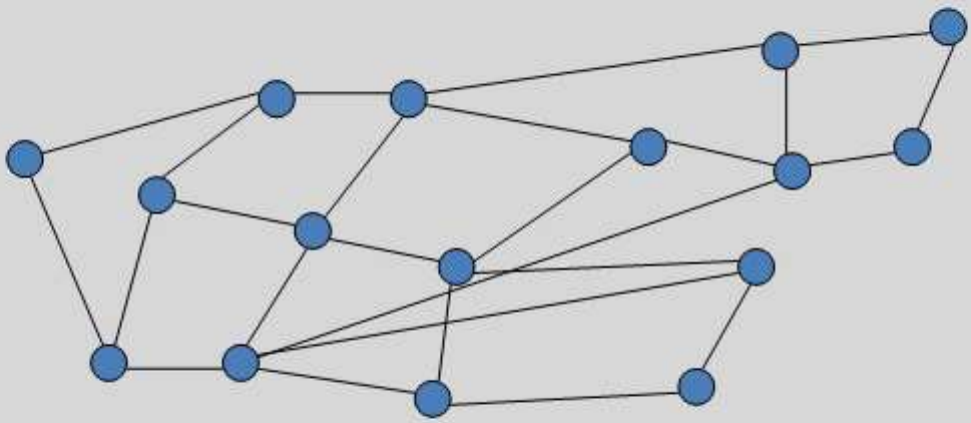
Can this graph be two colored?



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Algorithm

- Run BFS
- Color odd layers red, even layers blue
- If no edges between the same layer, the graph is bipartite
- If edge between two vertices of the same layer, then there is an odd cycle, and the graph is not bipartite

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Theorem: A graph is bipartite if and only if it has no odd cycles

Odd Length Cycle $\Rightarrow$ ¬ Bipartite.

No odd length cycle $\Rightarrow$ Bipartite.

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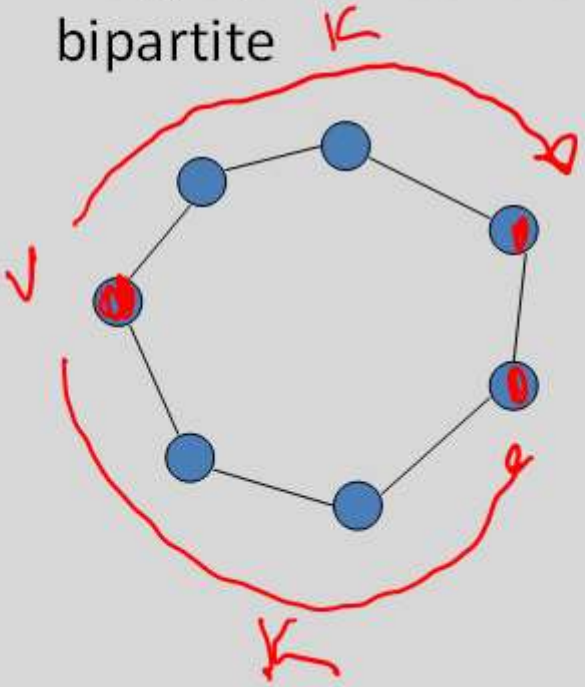
Lemma 1

- If a graph contains an odd cycle, it is not bipartite



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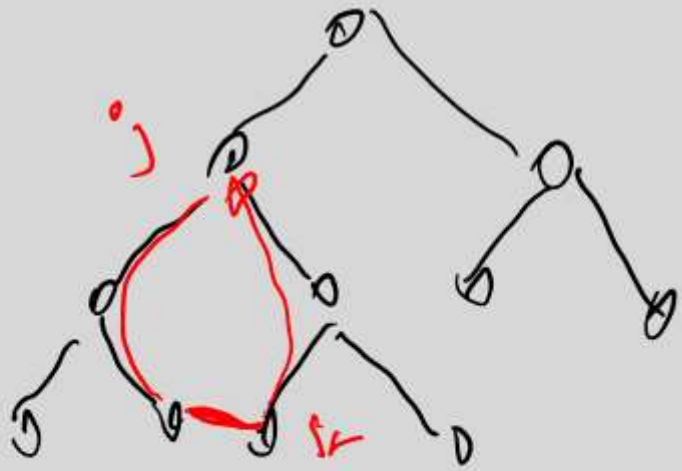
Lemma 2

- If a BFS tree has an *intra-level edge*, then the graph has an odd length cycle



Intra-level edge: both end points are in the same level

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Lemma 3

- If a graph has no odd length cycles, then it is bipartite

No odd cycles ⇒ no intra-level edges

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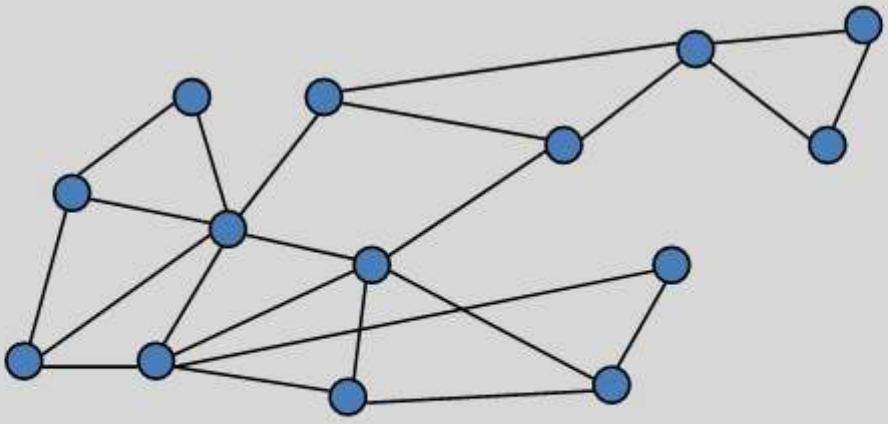
Graph Search

- Data structure for next vertex to visit determines search order



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Graph search

Breadth First Search

```
S = {s}
while S is not empty
    u = Dequeue(S)
    if u is unvisited
        visit u
        foreach v in N(u)
            Enqueue(S, v)
```

Depth First Search

```
S = {s}
while S is not empty
    u = Pop(S)
    if u is unvisited
        visit u
        foreach v in N(u)
            Push(S, v)
```

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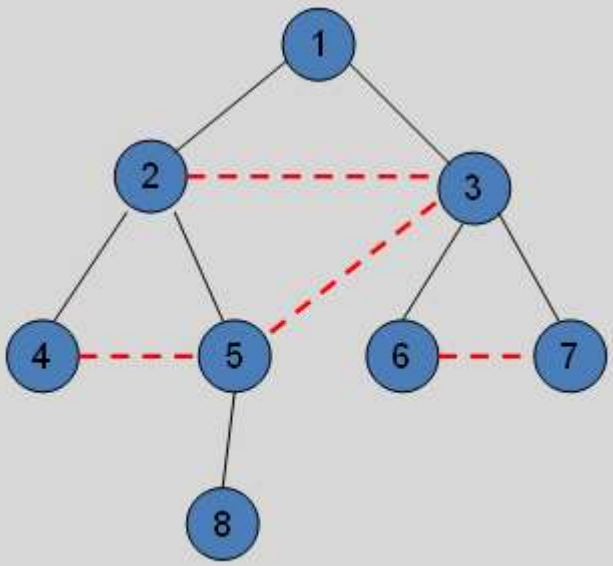
Breadth First Search

- All edges go between vertices on the same layer or adjacent layers



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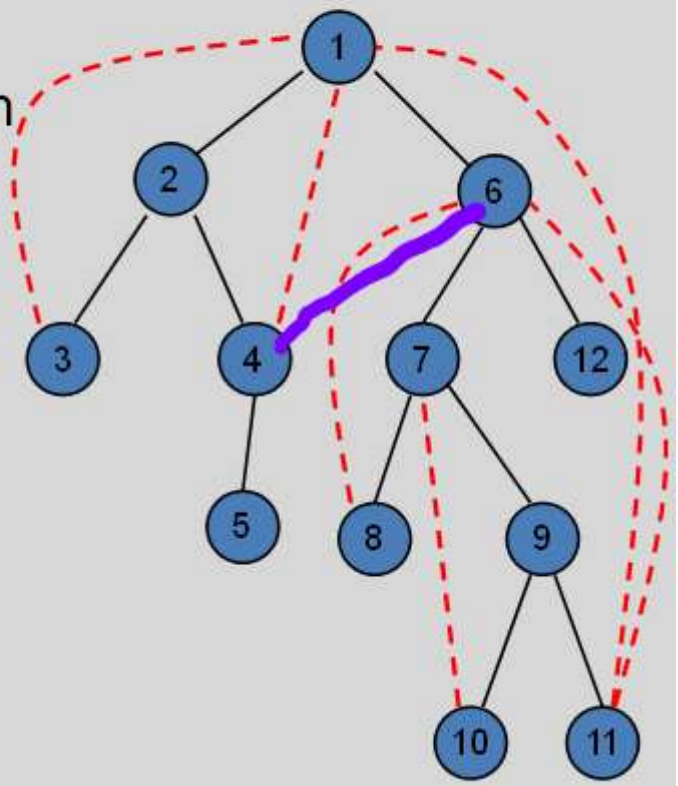
Depth First Search

- Each edge goes between vertices on the same branch
- No cross edges



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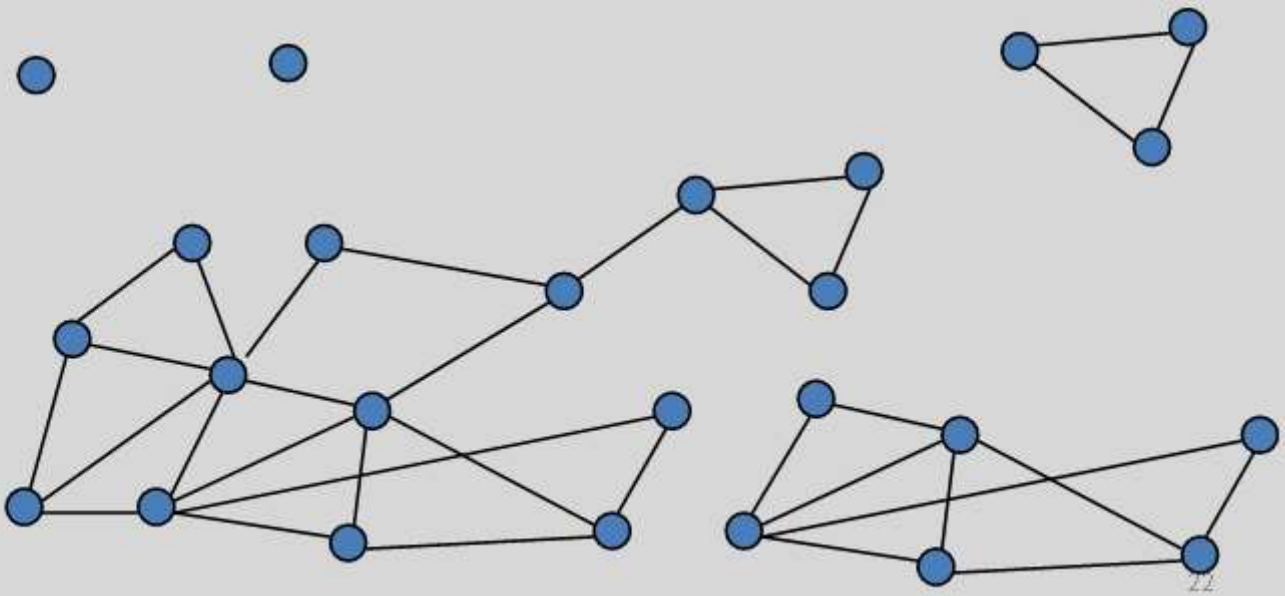
Connected Components

- Undirected Graphs



Computing Connected Components in O(n+m) time

- A search algorithm from a vertex v can find all vertices in v's component
- While there is an unvisited vertex v, search from v to find a new component

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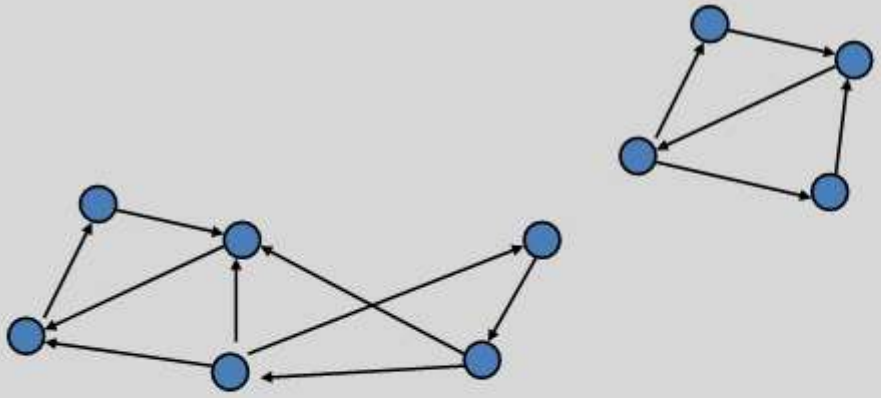
Directed Graphs

- A Strongly Connected Component is a subset of the vertices with paths between every pair of vertices.



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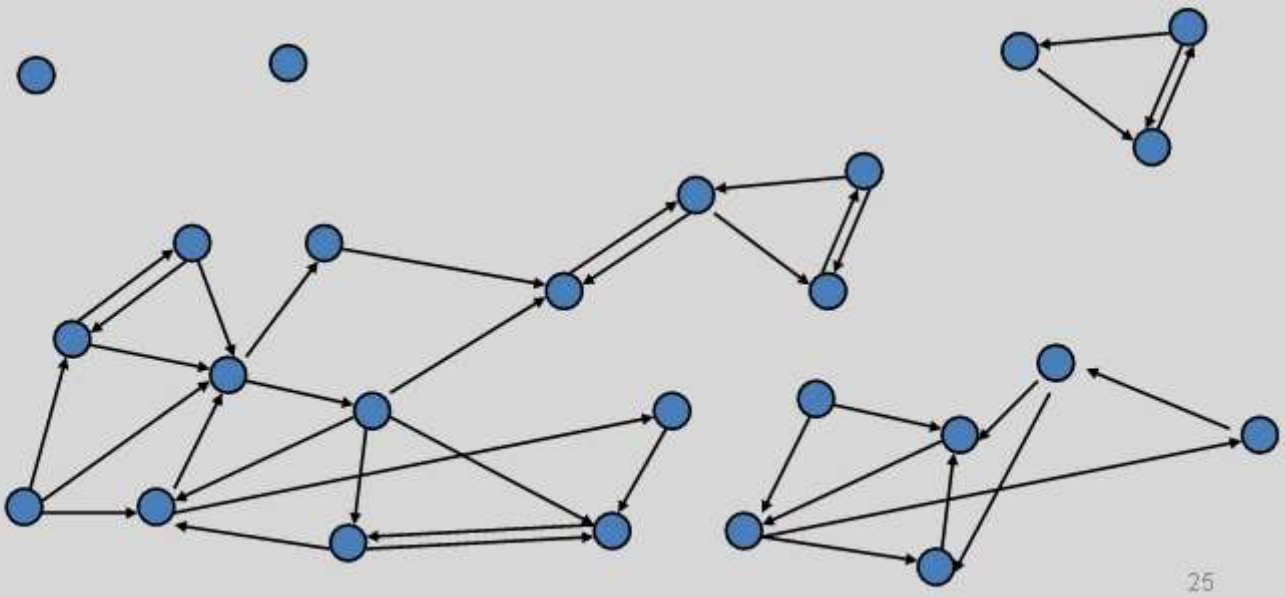
Identify the Strongly Connected Components



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