CSE 417
Algorithms and Complexity

Graphs and Graph Algorithms

Autumn 2024

Lecture 6

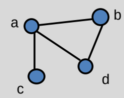
Announcements

- Reading
  - Chapter 3
  - Start on Chapter 4
- Homework 2

Graph Theory

- G = (V, E)
  - V: vertices, |V|= n
  - E: edges, |E| = m
- Undirected graphs
  - Edges sets of two vertices {u, v}
- Directed graphs
  - Edges ordered pairs (u, v)
- Many other flavors
  - Edge / vertices weights
  - Parallel edges
  - Self loops
- Path: $v_1, v_2, \ldots, v_k$, with (v_i, v_{i+1}) in E
  - Simple Path
  - Cycle
  - Simple Cycle
- Neighborhood
  - N(v)
- Distance
- Connectivity
  - Undirected
  - Directed (strong connectivity)
- Trees
  - Rooted
  - Unrooted

Graph Representation



V = { a, b, c, d}

E = { {a, b}, {a, c}, {a, d}, {b, d} }

a → b → c → d
b → a → d
c → a
d → a → b

Adjacency List

O(n + m) space

	1	1	1
1		0	1
1	0		0
1	1	0	

Incidence Matrix

O(n^2) space

Implementation Issues

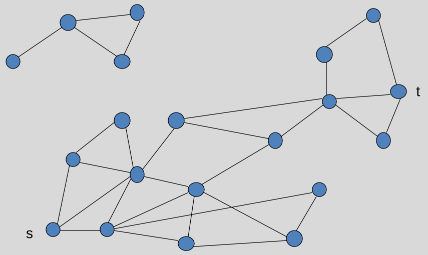
- Graph with n vertices, m edges
- Operations
  - Lookup edge
  - Add edge
  - Enumeration edges
  - Initialize graph
- Space requirements

Graph search

- Find a path from s to t

```
S = {s};  mark s
while S is not empty
        u = Select(S)
        if (u == t) then path found
        foreach v in N(u)
                if v is unmarked
                        mark v
                        Add(S, v)
                        Pred[v] = u
```

Graph Search



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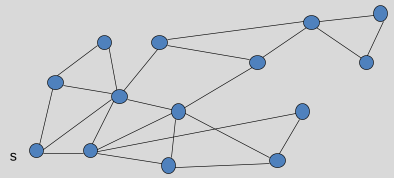
Breadth first search

- Explore vertices in layers
  - s in layer 1
  - Neighbors of s in layer 2
  - Neighbors of layer 2 in layer 3 . . .



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Breadth First Search

- Build a BFS tree from s

```
Initialize Level[v] = -1 for all v;
Q = {s}
Level[s] = 1;
while Q is not empty
        u = Q.Dequeue()
        foreach v in N(u)
                if (Level[v] == -1)
                        Q.Enqueue(v)
                        Pred[v] = u
                        Level[v] = Level[u] + 1
```

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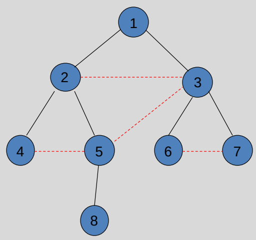
Key observation

- All edges go between vertices on the same layer or adjacent layers



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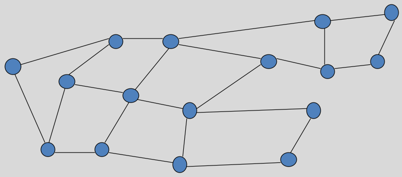
Bipartite Graphs

- A graph V is bipartite if V can be partitioned into V_1, V_2 such that all edges go between V_1 and V_2
- A graph is bipartite if it can be two colored



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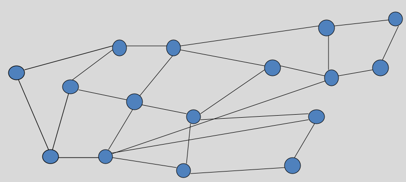
Can this graph be two colored?



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Algorithm

- Run BFS
- Color odd layers red, even layers blue
- If no edges between the same layer, the graph is bipartite
- If edge between two vertices of the same layer, then there is an odd cycle, and the graph is not bipartite

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Theorem: A graph is bipartite if and only if it has no odd cycles

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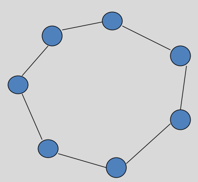
Lemma 1

- If a graph contains an odd cycle, it is not bipartite



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Lemma 2

- If a BFS tree has an *intra-level edge*, then the graph has an odd length cycle

Intra-level edge: both end points are in the same level

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Lemma 3

- If a graph has no odd length cycles, then it is bipartite

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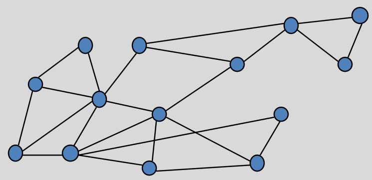
Graph Search

- Data structure for next vertex to visit determines search order



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Graph search

Breadth First Search

S = {s}

while S is not empty

  u = Dequeue(S)

  if u is unvisited

    visit u

    foreach v in N(u)

      Enqueue(S, v)

Depth First Search

S = {s}

while S is not empty

  u = Pop(S)

  if u is unvisited

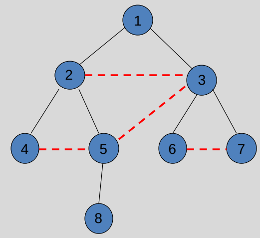
    visit u

    foreach v in N(u)

      Push(S, v)

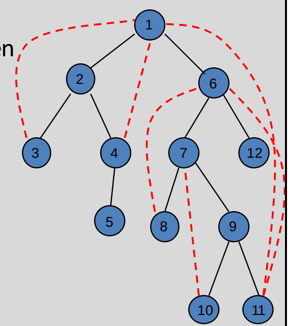
Breadth First Search

- All edges go between vertices on the same layer or adjacent layers



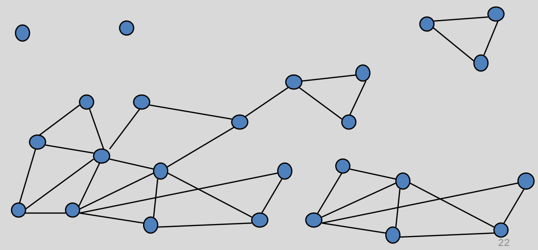
Depth First Search

- Each edge goes between vertices on the same branch
- No cross edges



Connected Components

- Undirected Graphs

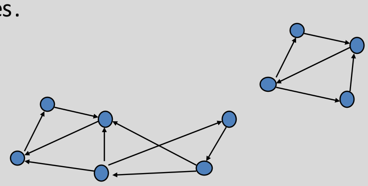


Computing Connected Components in O(n+m) time

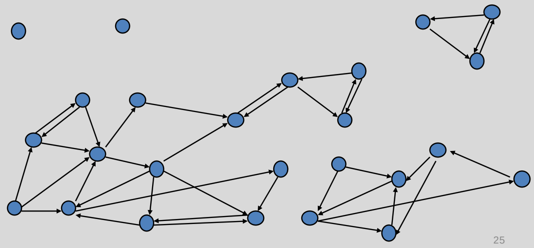
- A search algorithm from a vertex v can find all vertices in v's component
- While there is an unvisited vertex v, search from v to find a new component

Directed Graphs

- A Strongly Connected Component is a subset of the vertices with paths between every pair of vertices.



Identify the Strongly Connected Components



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