CSE 417
Algorithms and Complexity
Winter 2023
Lecture 25
NP-Completeness, Part III
Announcements

- Homework 9
- Exam practice problems on course homepage
- Final Exam: Monday, March 13, 8:30 AM

<table>
<thead>
<tr>
<th>Date</th>
<th>Topic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fri, March 3</td>
<td>NP-Completeness: Overview, Definitions</td>
</tr>
<tr>
<td>Mon, March 6</td>
<td>NP-Completeness: Reductions</td>
</tr>
<tr>
<td>Wed, March 8</td>
<td>NP-Completeness: Problem Survey</td>
</tr>
<tr>
<td>Fri, March 10</td>
<td>Theory and Beyond NP-Completeness</td>
</tr>
<tr>
<td>Mon, March 13</td>
<td>Final Exam</td>
</tr>
</tbody>
</table>
NP Completeness: The story so far

Circuit Satisfiability is NP-Complete

There are a whole bunch of other important problems which are NP-Complete
Cook’s Theorem

• Definition:
  – X is NP-Complete if:
    • X is in NP
    • For all Z in NP: \( Z \leq_P X \)

• There is an NP Complete problem
  – The Circuit Satisfiability Problem
Populating the NP-Completeness Universe

- Circuit Sat $\leq_p$ 3-SAT
- 3-SAT $\leq_p$ Independent Set
- 3-SAT $\leq_p$ Vertex Cover
- Independent Set $\leq_p$ Clique
- 3-SAT $\leq_p$ Hamiltonian Circuit
- Hamiltonian Circuit $\leq_p$ Traveling Salesman
- 3-SAT $\leq_p$ Integer Linear Programming
- 3-SAT $\leq_p$ Graph Coloring
- 3-SAT $\leq_p$ 3 Dimensional Matching
- 3-SAT $\leq_p$ Subset Sum
- Subset Sum $\leq_p$ Scheduling with Release times and deadlines
Satisfiability

Literal: A Boolean variable or its negation.

Clause: A disjunction of literals.

Conjunctive normal form: A propositional formula \( \Phi \) that is the conjunction of clauses.

SAT: Given CNF formula \( \Phi \), does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: \( (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \)

Yes: \( x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false} \).
Matching

Two dimensional matching

Three dimensional matching (3DM)
Augmenting Path Algorithm for Matching

Find augmenting path in $O(m)$ time
$n$ phases of augmentation
$O(nm)$ time algorithm for matching
3-SAT $\leq_p$ 3DM

Truth Setting Gadget

X True

X False

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3-SAT $\leq_p$ 3DM

Clause gadget for ($\overline{X}$ OR $Y$ OR $Z$)

Garbage Collection Gadget
(Many copies)
Exact Cover (sets of size 3) XC3

Given a collection of sets of size 3 of a domain of size 3N, is there a sub-collection of N sets that cover the sets

(A, B, C), (D, E, F), (A, B, G),
(A, C, I), (B, E, G), (A, G, I),
(B, D, F), (C, E, I), (C, D, H),
(D, G, I), (D, F, H), (E, H, I),
(F, G, H), (F, H, I)

3DM $\leq_p$ XC3
Graph Coloring

- NP-Complete
  - Graph K-coloring
  - Graph 3-coloring

- Polynomial
  - Graph 2-Coloring
3-SAT $\leq_p$ 3 Colorability

Truth Setting Gadget

Clause Testing Gadget
(Can be colored if at least one input is T)
Number Problems

• Subset sum problem
  – Given natural numbers $w_1, \ldots, w_n$ and a target number $W$, is there a subset that adds up to exactly $W$?

• Subset sum problem is NP-Complete
• Subset Sum problem can be solved in $O(nW)$ time
XC3 $\leq_p$ SUBSET SUM

Idea: Represent each set as a large integer, where the element $x_i$ is encoded as $D^i$ where $D$ is an integer

$$\{x_3, x_5, x_9\} \Rightarrow D^3 + D^5 + D^9$$

Does there exist a subset that sums to exactly $D^1 + D^2 + D^3 + \ldots + D^{n-1} + D^n$

Detail: How large is $D$? We need to make sure that we do not have any carries, so we can choose $D = m+1$, where $m$ is the number of sets.
Integer Linear Programming

- Linear Programming – maximize a linear function subject to linear constraints
- Integer Linear Programming – require an integer solution
- NP Completeness reduction from 3-SAT

Use 0-1 variables for $x_i$’s

Constraint for clause: $\left(x_1 \lor \overline{x_2} \lor \overline{x_2}\right)$

$$x_1 + (1 - x_2) + (1-x_3) > 0$$
Scheduling with release times and deadlines (RD-Sched)

• Tasks, \{t_1, t_2, \ldots, t_n\}
• Task \( t_j \) has a length \( l_j \), release time \( r_j \) and deadline \( d_j \)
• Once a task is started, it is worked on without interruption until it is completed
• Can all tasks be completed satisfying constraints?
Subset Sum $\leq_P$ RD-Sched

- Subset Sum Problem
  - $\{s_1, s_2, \ldots, s_N\}$, integer $K_1$
  - Does there exist a subset that sums to $K_1$?
  - Assume the total sums to $K_2$
Reduction

- Tasks \( \{ t_1, t_2, \ldots, t_N, x \} \)
- \( t_j \) has length \( s_j \), release 0, deadline \( K_2 + 1 \)
- \( x \) has length 1, release \( K_1 \), deadline \( K_1 + 1 \)
Friday: NP-Completeness and Beyond!