CSE 417
Algorithms and Complexity
Winter 2023
Lecture 24
NP-Completeness, Part II
Announcements

• Homework 9
• Exam practice problems on course homepage
• Final Exam: Monday, March 13, 8:30 AM

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Key Idea: Problem Reduction

• Use an algorithm for problem X to solve problem Y.
  – This means that problem X is more difficult than problem Y
• Terminology: Y is reducible to X
  – Notation: $Y \leq_p X$
The Universe

- **P**: Polynomial Time
- **NP**: Nondeterministic Polynomial Time
  - Problems where a “yes” answer can be verified in polynomial time
- **NP-Complete**
  - The hardest problems in NP
NP-Completeness

• If \( X \) is NP-Complete, \( Z \) is in NP and \( X \prec_P Z \)
  – \( Z \) is NP-Complete

• Steve Cook got this started by finding the first NP-Complete problem
3-Coloring $\leq_p$ 4-Coloring

3-Coloring Problem

Convert to a 4-Coloring Problem

Solve 4-Coloring Problem

Convert Solution back to a 3-Coloring
Starting NP-Complete Problems

• Circuit Satisfiability Problem
  – Cook’s Theorem
• Boolean Formula Satisfiability
  – 3-SAT
• Maximum Independent Set
Circuit SAT

Find a satisfying assignment
Satisfiability

Literal: A Boolean variable or its negation. \( x_i \) or \( \overline{x_i} \)

Clause: A disjunction of literals. \( C_j = x_1 \lor \overline{x_2} \lor x_3 \)

Conjunctive normal form: A propositional formula \( \Phi \) that is the conjunction of clauses. \( \Phi = C_1 \land C_2 \land C_3 \land C_4 \)

SAT: Given CNF formula \( \Phi \), does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: \( (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \)

Yes: \( x_1 = \text{true}, \ x_2 = \text{true} \ x_3 = \text{false} \).
3-SAT is NP-Complete

Circuit SAT $\leq_P$ 3-SAT

Convert a circuit into a formula

Each gate is represented by a set of clauses

Output = 1

\[
\begin{align*}
(x_1 \vee x_1 \vee x_1) \land (x_2 \vee x_2 \vee x_3) \land \\
(\overline{x_2} \vee \overline{x_2} \vee \overline{x_3}) \land (x_1 \vee x_1 \vee \overline{x_4}) \land \\
(x_1 \vee x_1 \vee \overline{x_5}) \land (\overline{x_1} \vee x_4 \vee x_5) \land \\
(\overline{x_0} \vee \overline{x_0} \vee x_1) \land (\overline{x_0} \vee \overline{x_0} \vee x_2) \land \\
(x_0 \vee \overline{x_1} \vee \overline{x_2})
\end{align*}
\]
Independent Set

- Independent Set
  - Graph $G = (V, E)$, a subset $S$ of the vertices is independent if there are no edges between vertices in $S$
3 Satisfiability Reduces to Independent Set

Claim. 3-SAT ≤_P INDEPENDENT-SET.

Pf. Given an instance Φ of 3-SAT, we construct an instance (G, k) of INDEPENDENT-SET that has an independent set of size k iff Φ is satisfiable.

Construction.

– G contains 3 vertices for each clause, one for each literal.
– Connect 3 literals in a clause in a triangle.
– Connect literal to each of its negations.

\[
\Phi = (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_4)
\]
Vertex Cover

- **Vertex Cover**
  - Graph $G = (V, E)$, a subset $S$ of the vertices is a vertex cover if every edge in $E$ has at least one endpoint in $S$.
IS $\leq_{p} VC$

• Lemma: A set $S$ is independent iff $V - S$ is a vertex cover

• To reduce IS to VC, we show that we can determine if a graph has an independent set of size $K$ by testing for a Vertex cover of size $n - K$
IS $\leq^p$ VC

Find a maximum independent set $S$

Show that $V-S$ is a vertex cover
Clique

- Clique
  - Graph $G = (V, E)$, a subset $S$ of the vertices is a clique if there is an edge between every pair of vertices in $S$
Complement of a Graph

- Defn: $G'=(V,E')$ is the complement of $G=(V,E)$ if $(u,v)$ is in $E'$ iff $(u,v)$ is not in $E$.
Lemma: S is Independent in G iff S is a Clique in the complement of G

To reduce IS to Clique, we compute the complement of the graph. The complement has a clique of size K iff the original graph has an independent set of size K.
Hamiltonian Circuit Problem

- Hamiltonian Circuit – a simple cycle including all the vertices of the graph
Thm: Hamiltonian Circuit is NP Complete

• Reduction from 3-SAT
Clause Gadget

\[ x_1 \lor x_2 \lor x_3 \]

X₁ Group

X₂ Group

X₃ Group
Hamiltonian Path Problem

- Hamiltonian Path – a simple path including all the vertices of the graph
Reduce Hamiltonian Circuit to Hamiltonian Path

$G_2$ has a Hamiltonian Path iff $G_1$ has a Hamiltonian Circuit
Traveling Salesman Problem

• Given a complete graph with edge weights, determine the shortest tour that includes all of the vertices (visit each vertex exactly once, and get back to the starting point)

Find the minimum cost tour
Thm: HC $\lesssim_p$ TSP