Announcements

- Homework 8, Due Wednesday, Nov 29
- Homework 9, Due Friday, Dec 8

Network Flow

Outline

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem

Network Flow Definitions

- Capacity
- Source, Sink
- Capacity Condition
- Conservation Condition
- Value of a flow

Flow Example
Network Flow Definitions

- **Flowgraph**: Directed graph with distinguished vertices \( s \) (source) and \( t \) (sink)
- **Capacities on the edges**, \( c(e) \geq 0 \)
- **Problem**, assign flows \( f(e) \) to the edges such that:
  - \( 0 \leq f(e) \leq c(e) \)
  - Flow is conserved at vertices other than \( s \) and \( t \)
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is as large as possible

Find a maximum flow

Residual Graph

- **Flow graph showing the remaining capacity**
- **Flow graph** \( G \), **Residual Graph** \( G_R \)
  - \( G \): edge \( e \) from \( u \) to \( v \) with capacity \( c \) and flow \( f \)
  - \( G_R \): edge \( e' \) from \( u \) to \( v \) with capacity \( c - f \)
  - \( G_R \): edge \( e'' \) from \( v \) to \( u \) with capacity \( f \)
Augmenting Path Algorithm

- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
  - $v_1 = s$, $v_k = t$
  - Possible to add $b$ units of flow between $v_j$ and $v_{j+1}$ for $j = 1 \ldots k - 1$

Find two augmenting paths

Augmenting Path Lemma

- Let $P = v_1, v_2, \ldots, v_k$ be a path from $s$ to $t$ with minimum capacity $b$ in the residual graph.
- $b$ units of flow can be added along the path $P$ in the flow graph.

Proof

- Add $b$ units of flow along the path $P$
  - What do we need to verify to show we have a valid flow after we do this?

Ford-Fulkerson Algorithm (1956)

while not done
  Construct residual graph $G_R$
  Find an $s$-$t$ path $P$ in $G_R$ with capacity $b > 0$
  Add $b$ units along in $G$

If the sum of the capacities of edges leaving $S$ is at most $C$, then the algorithm takes at most $C$ iterations