CSE 417: Algorithms and Computational Complexity

Lecture 2: Analysis

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Algorithm "efficiency" will pervade the course; what is it, how do we quantify it? "Big-O"!

Why big-O: measuring algorithm efficiency

What's big-O: definition and related concepts

Reasoning with big-O: examples & applications

- polynomials
- exponentials
- logarithms
- sums

Polynomial Time

Why big-O: measuring algorithm efficiency

What is the n^{th} prime number?

Let p_n = n^{th} prime, n >= 1, e.g.:

$p_1 = 2$

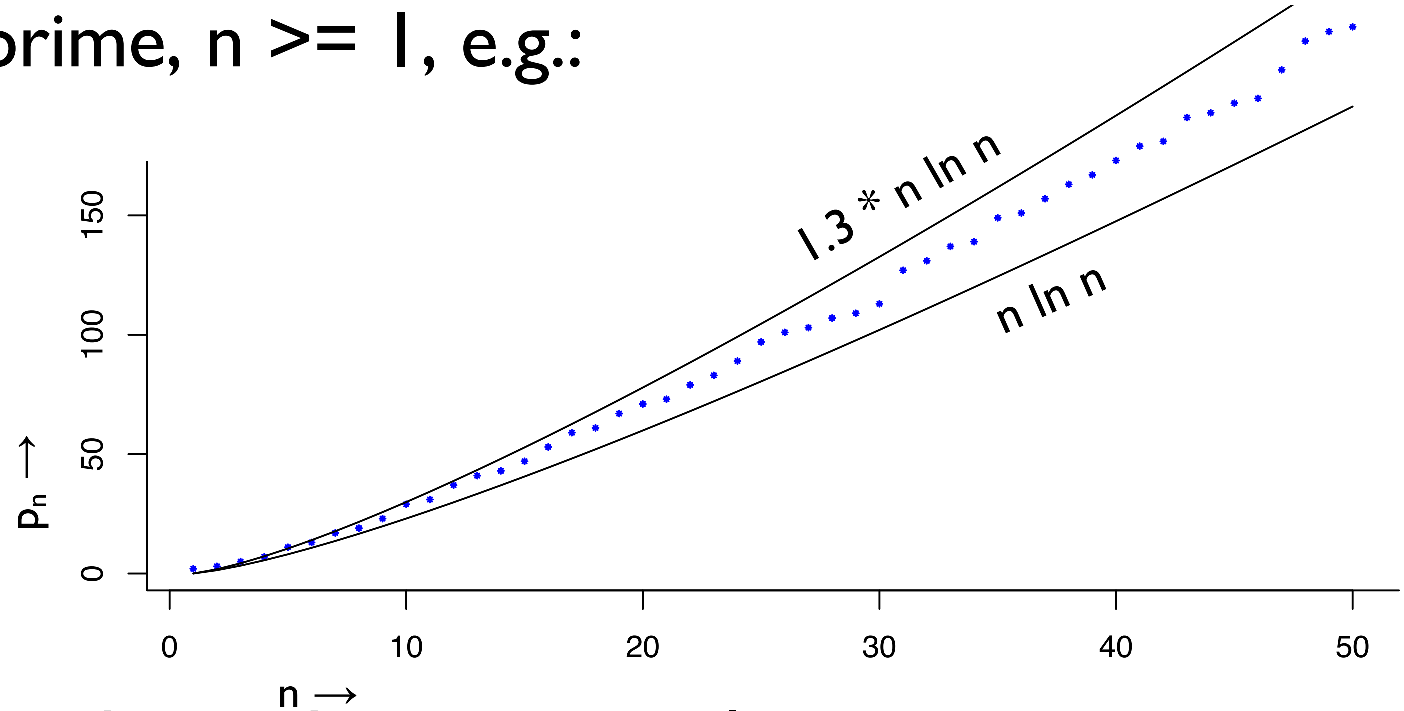
$p_2 = 3$

$p_3 = 5$

$p_4 = 7$

$p_5 = 11$

$P_6 = 13$



After much study, we know $p_n \sim n \log n$; even:

$$\frac{p_n}{n} = \log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n} - \frac{(\log\log n)^2 - 6\log\log n + 11}{2(\log n)^2} + o\left(\frac{1}{(\log n)^2}\right)$$

Great precision! *But,* often a simple, smooth, upper bound, is more convenient, e.g.: $p_n = O(n \log n)$

https://en.wikipedia.org/wiki/Prime_number_theorem

Our correct TSP algorithm was incredibly slow

No matter what computer you have

As a 2nd example, for large problems, mergesort beats insertion sort – n log n *vs* n^2 matters a lot

No matter what computer you have

Even tho m-sort is more complex & inner loop is slower

We want a general theory of “efficiency” that is

Simple

Objective

Relatively independent of changing technology

Measures *algorithm*, not code

But still *predictive* – “theoretically bad” algorithms should be bad in practice and vice versa (usually)

And very complex fns

The *time complexity* of an algorithm associates a number T(n), the worst-case time the algorithm takes, with each problem size n.

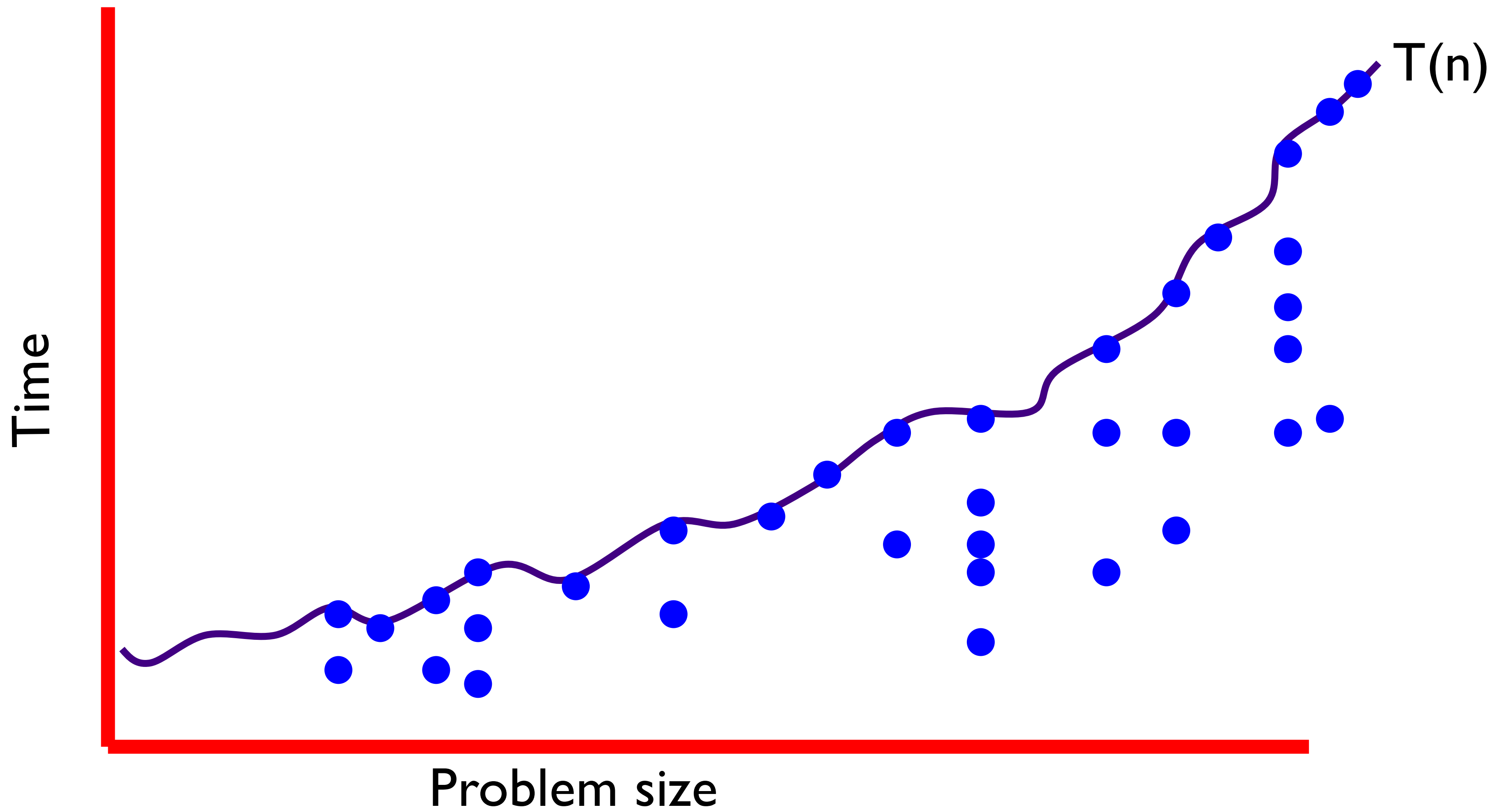
Mathematically,

T: N+ $\rightarrow$ R

i.e., T is a function mapping positive integers (problem sizes) to positive real numbers (number of steps).

"Reals" so, e.g., we can say sqrt(n) instead of $\lceil$sqrt(n)$\rceil$

"Positive" so, e.g., log(n) and 2^n/n aren't problematic


T(n)
Time
Problem size

Asymptotic growth rate: i.e., characterize growth rate of worst-case run time as a function of problem size, up to a constant factor, e.g. $T(n) = O(n^2)$

Why not try to be more precise?

Average-case, e.g., is hard to define, analyze, maybe misleading

Technological variations (computer, compiler, OS, …) easily 10x or more

Being more precise is *much* more work

A key question is "*scale up*": if I can afford this today, how much longer will it take when my business is 2x larger? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.)

Big-O analysis is adequate to address this.

NOT saying c=10 vs c=2 is irrelevant; it's secondary.

What's big-O: definition and related concepts

Given two functions f and g: N+ $\rightarrow$ R

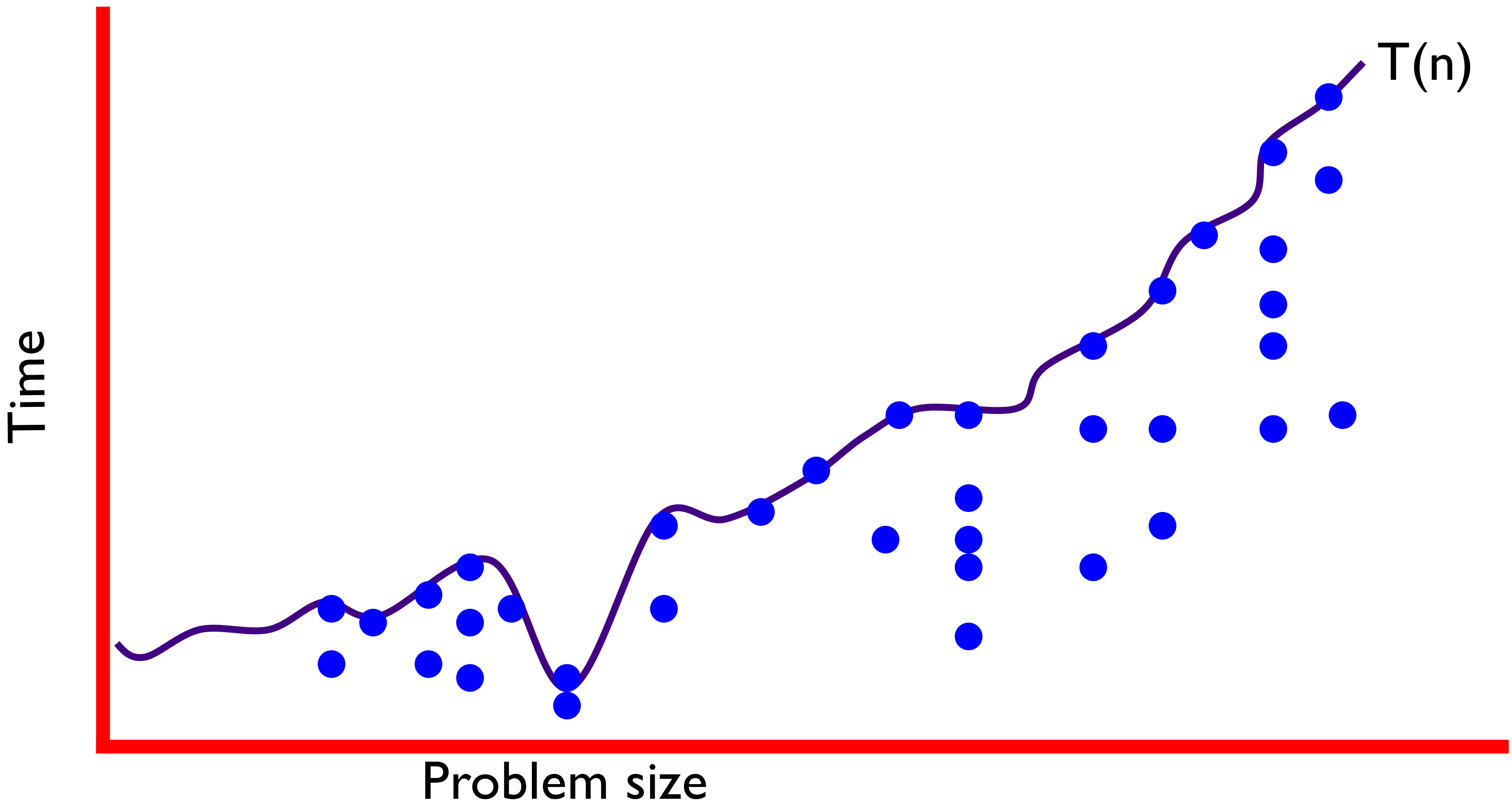
f(n) is O(g(n)) iff there is a constant c > 0 so that f(n) is eventually always $\leq$ c g(n) — Upper Bounds

f(n) is Ω(g(n)) iff there is a constant c > 0 so that f(n) is eventually always $\geq$ c g(n) — Lower Bounds

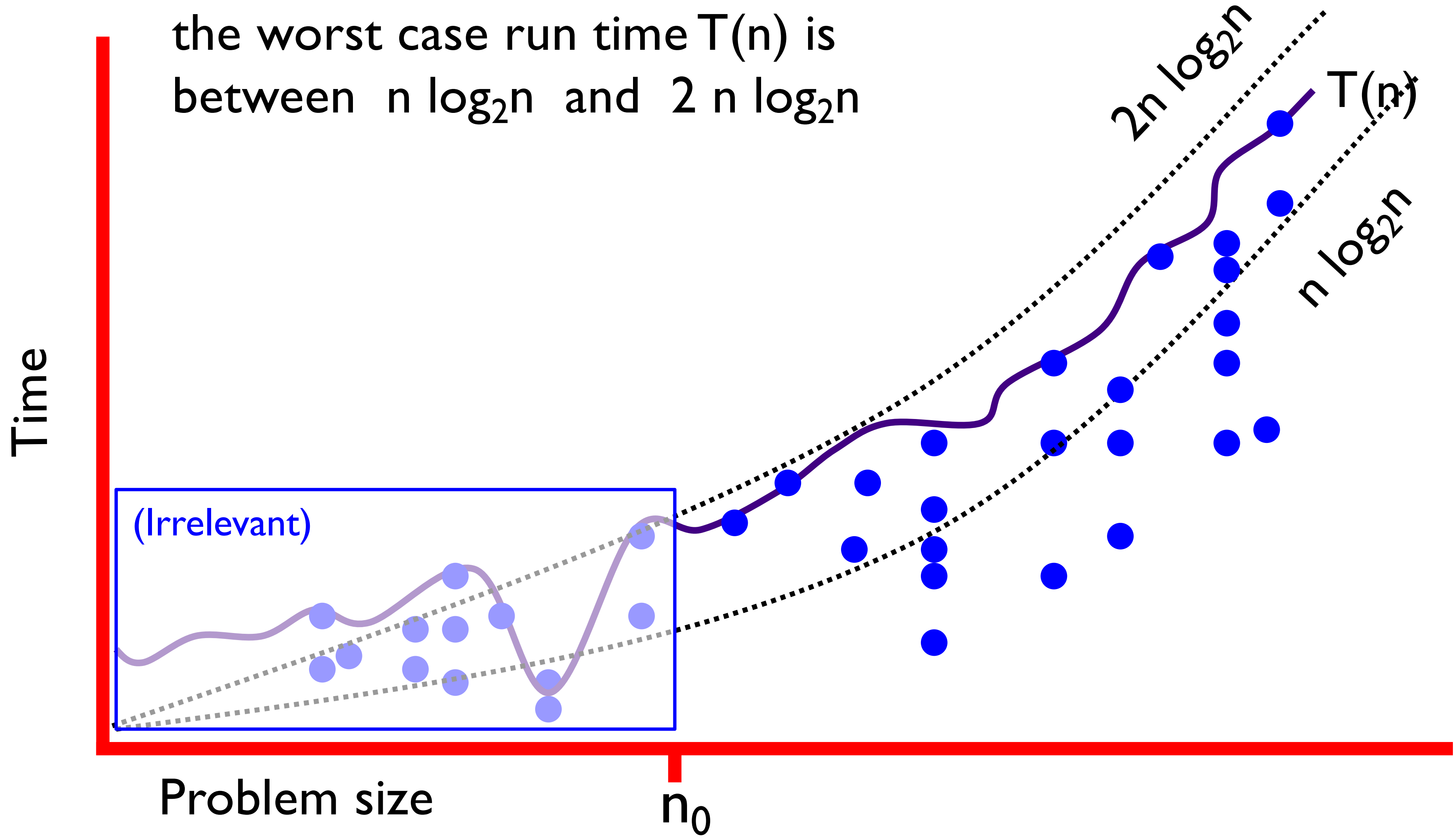
f(n) is Θ(g(n)) iff there is are constants c_1, c_2 > 0 so that eventually always $c_1 g(n) \leq f(n) \leq c_2 g(n)$ — Both

"Eventually always P(n)" means "P(n) is true" except perhaps for finitely many "small" values of n. Formally: "$\exists n_0$ s.t.$\forall n>n_0$ P(n) is true."

For out applications, f(n) is the (messy) actual time complexity of an algorithm, whereas g(n) is a simple approximation to it.


T(n)
Time
Problem size

Example: $T(n) = \Theta(n \log n)$
since for all problem sizes $n > n_0$,
the worst case run time $T(n)$ is
between $n \log_2 n$ and $2 n \log_2 n$



A program with initialization and two nested loops

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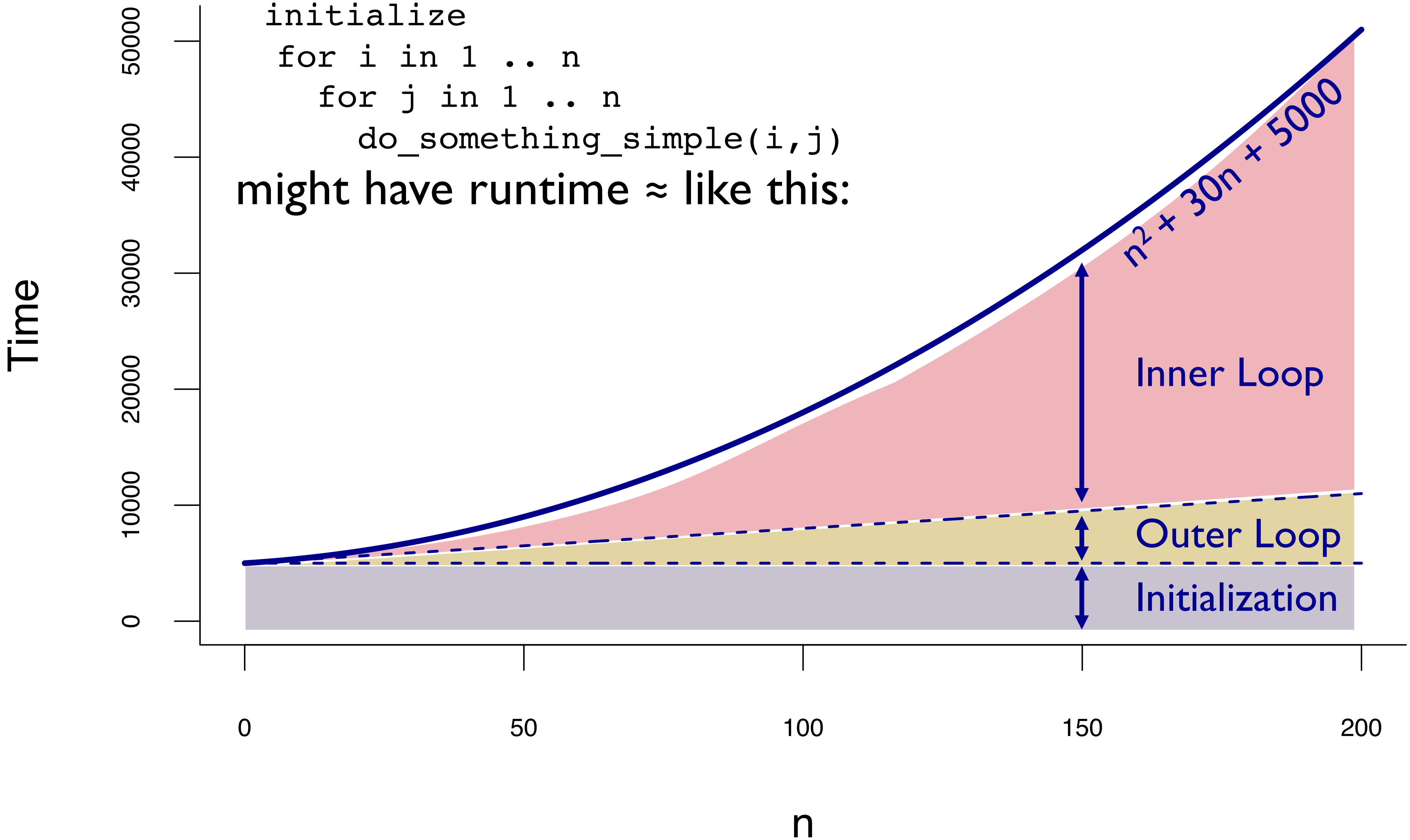
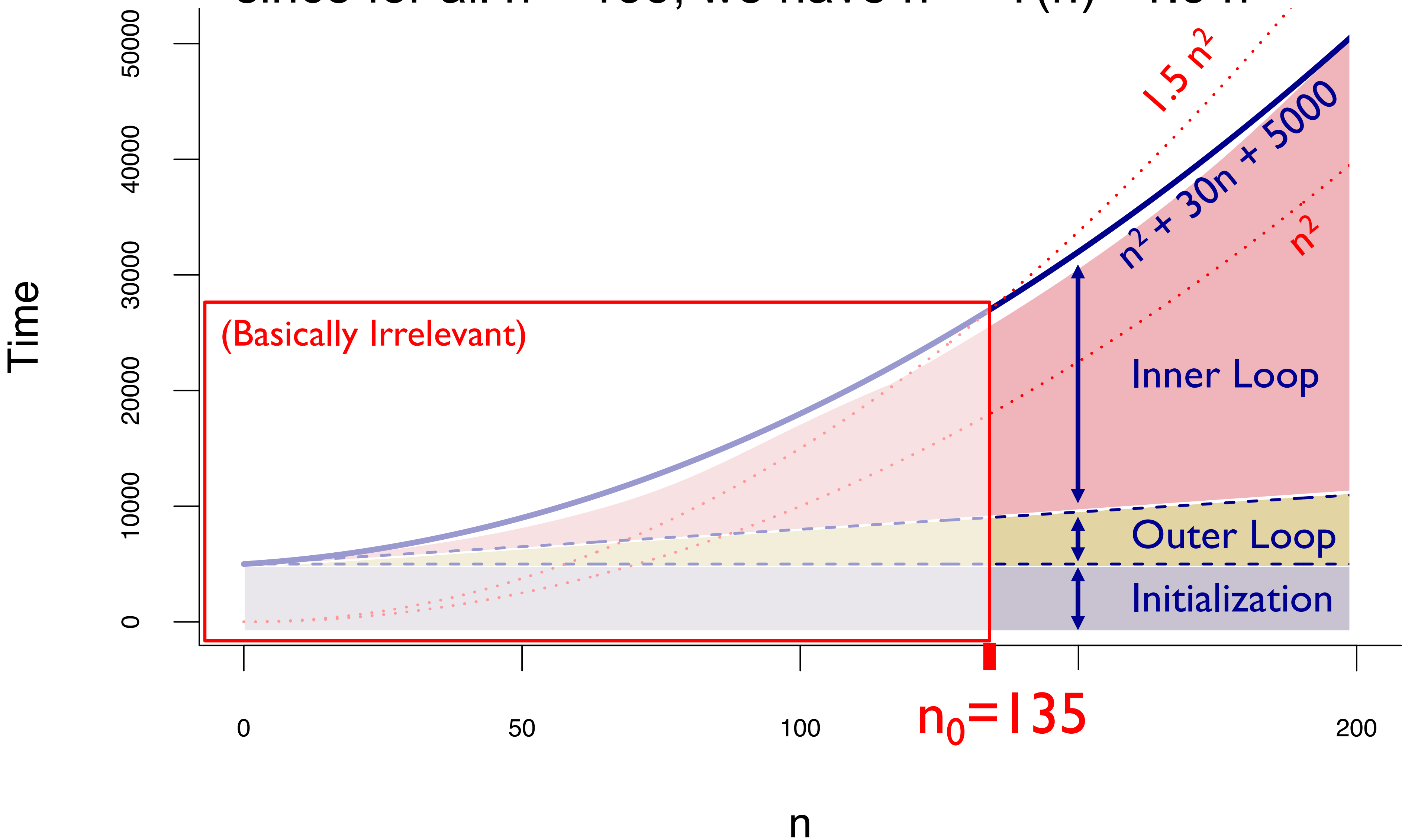
initialize
 for i in 1 .. n
   for j in 1 .. n
     do_something_simple(i,j)
```

might have runtime ≈ like this:



If $T(n) = n^2 + 30n + 5000$, then $T(n) = \Theta(n^2)$,
since for all $n \geq 135$, we have $n^2 \leq T(n) \leq 1.5\, n^2$



Asymptotic Notation in Seven Words

$$\textit{suppress}\ \underbrace{\textit{constant factors}}_{\textit{too system-dependent}}\ \textit{and}\ \underbrace{\textit{lower-order terms}}_{\textit{irrelevant for large inputs}}$$

Source: Tim Roughgarden's Algorithms book

Reasoning with big-O: examples & applications

polynomials

exponentials

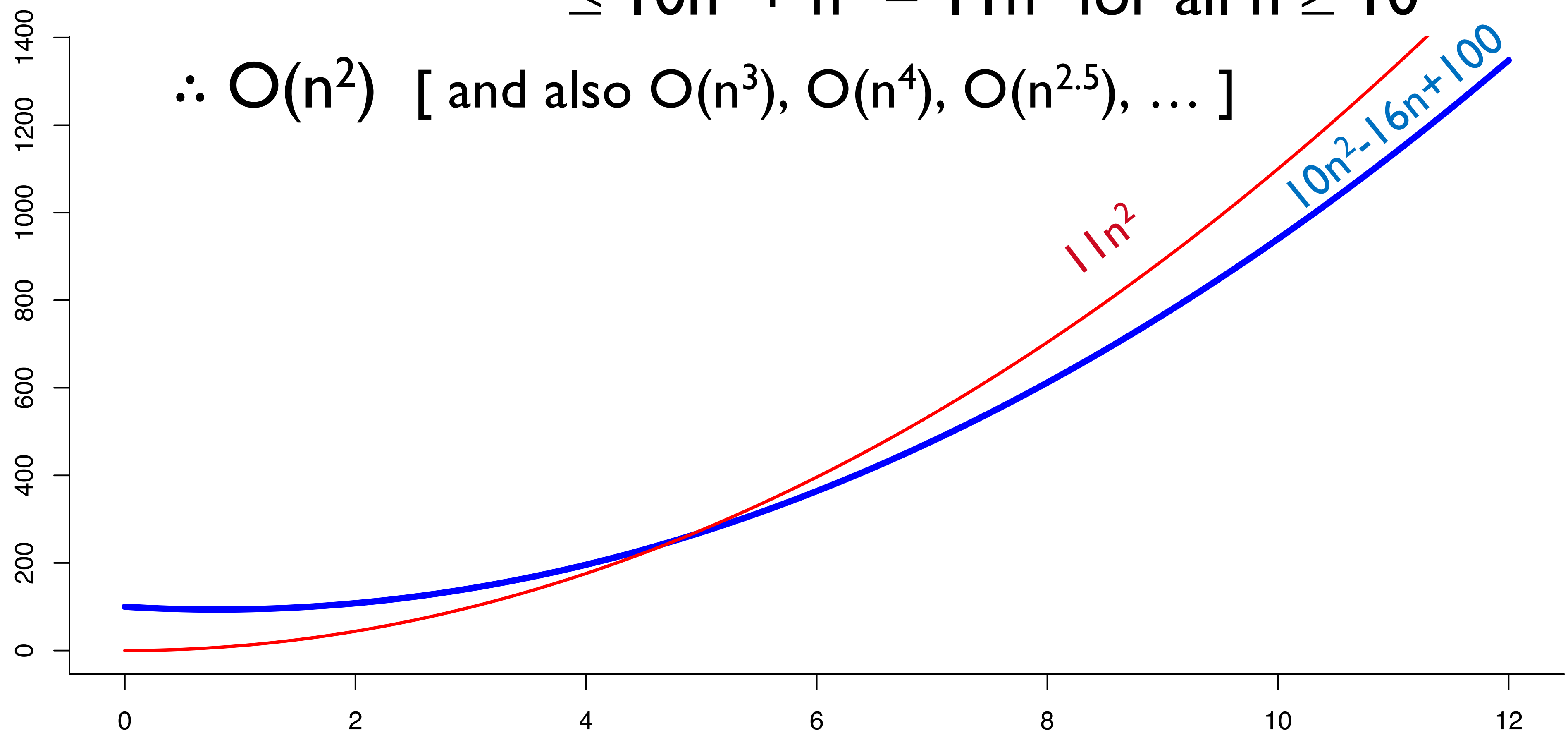
logarithms

sums

Show $10n^2-16n+100$ is $O(n^2)$:

$$10n^2-16n+100 \le 10n^2 + 100$$
$$= 10n^2 + 10^2$$
$$\le 10n^2 + n^2 = 11n^2 \text{ for all } n \ge 10$$

$\therefore O(n^2)$ [and also $O(n^3)$, $O(n^4)$, $O(n^{2.5})$, …]



Show $10n^2$-$16n+100$ is $\Omega(n^2)$:

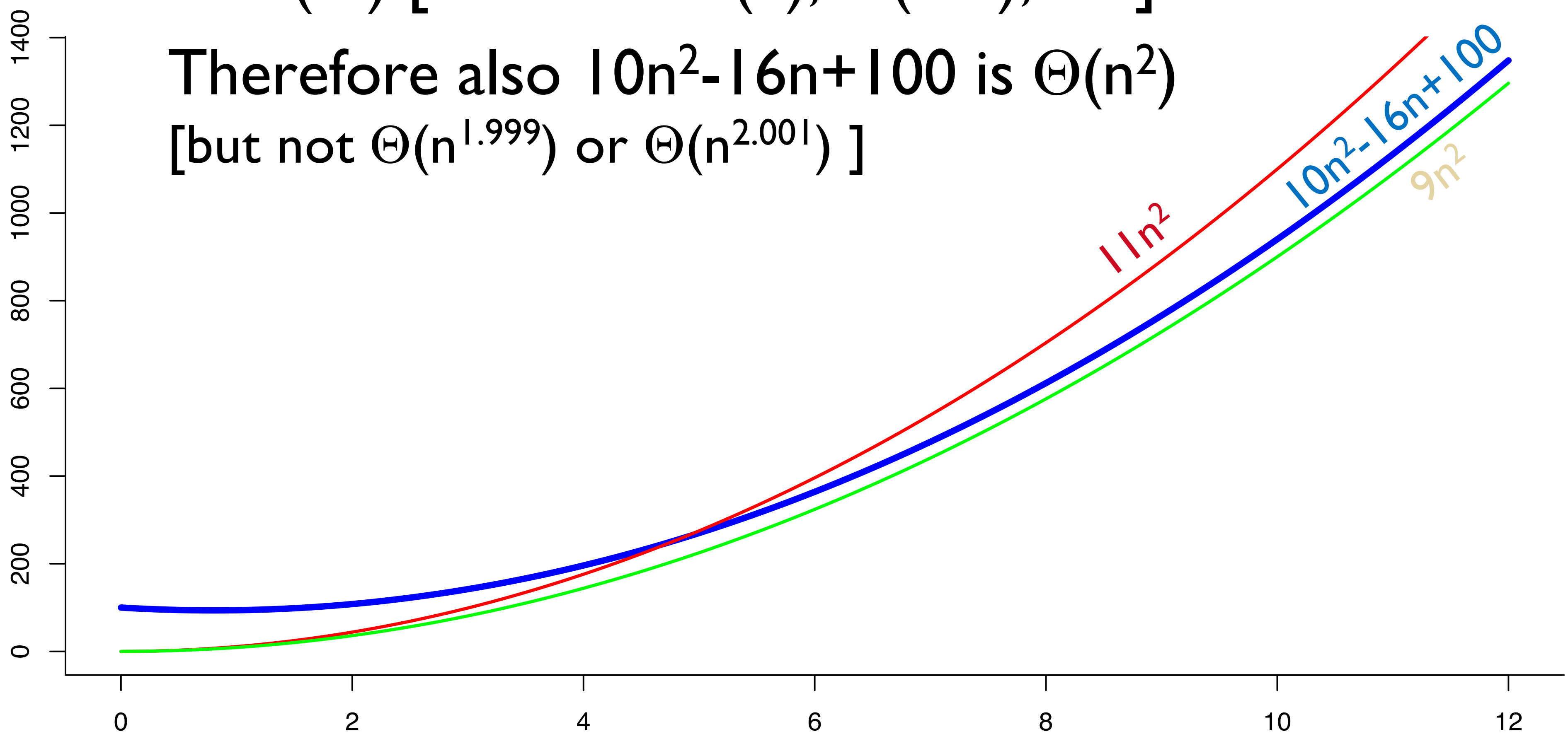
$10n^2 - 16n + 100 \geq 10n^2 - 16n$

$\geq 10n^2 - n^2 = 9n^2$ for all $n \geq 16$

$\therefore\ \Omega(n^2)$ [and also $\Omega(n)$, $\Omega(n^{1.5})$, …]

Therefore also $10n^2$-$16n+100$ is $\Theta(n^2)$

[but not $\Theta(n^{1.999})$ or $\Theta(n^{2.001})$]



Polynomials:

$p(n) = a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$

Proof:

$$\begin{aligned} p(n) &= a_0 + a_1 n + \ldots + a_d n^d \\ &\le |a_0| + |a_1| n + \ldots + a_d n^d \\ &\le |a_0| n^d + |a_1| n^d + \ldots + a_d n^d \qquad \text{(for } n \ge 1\text{)} \\ &= c\, n^d, \text{ where } c = (|a_0| + |a_1| + \ldots + |a_{d-1}| + a_d) \end{aligned}$$

$\therefore p(n) = O(n^d)$

Exercise: show that $p(n) = \Omega(n^d)$

Hint: this direction is trickier; focus on the "worst case" where all coefficients except a_d are negative.

another example of working with O-Ω-Θ notation

Example: For any a, and any $b > 0$, $(n+a)^b$ is $\Theta(n^b)$

$(n+a)^b \le (2n)^b$ for $n \ge |a|$

$= 2^b n^b$

$= cn^b$ for $c = 2^b$

so $(n+a)^b$ is $O(n^b)$

$(n+a)^b \ge (n/2)^b$ for $n \ge 2|a|$ (even if $a < 0$)

$= 2^{-b} n^b$

$= c'n$ for $c' = 2^{-b}$

so $(n+a)^b$ is $\Omega(n^b)$

more examples: tricks for sums

Example: $\sum_{1 \le i \le n} i = \Theta(n^2)$

```
E.g.:  for i = 1..n {
         for j=1 to i {
           ...
       }}
```

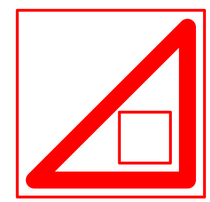
Proof:

(a) An upper bound: each term is ≤ the max term

$$\sum_{1 \le i \le n} i \le \sum_{1 \le i \le n} n = n^2 = O(n^2)$$

(b) A lower bound: each term is ≥ the min term

$$\sum_{1 \le i \le n} i \ge \sum_{1 \le i \le n} 1 = n = \Omega(n)$$



This is valid, but a weak bound.

Better: pick a large subset of large terms

$$\sum_{1 \le i \le n} i \ge \sum_{n/2 \le i \le n} n/2 \ge \lfloor n/2 \rfloor^2 = \Omega(n^2)$$

Transitivity.

If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.

If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.

If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.

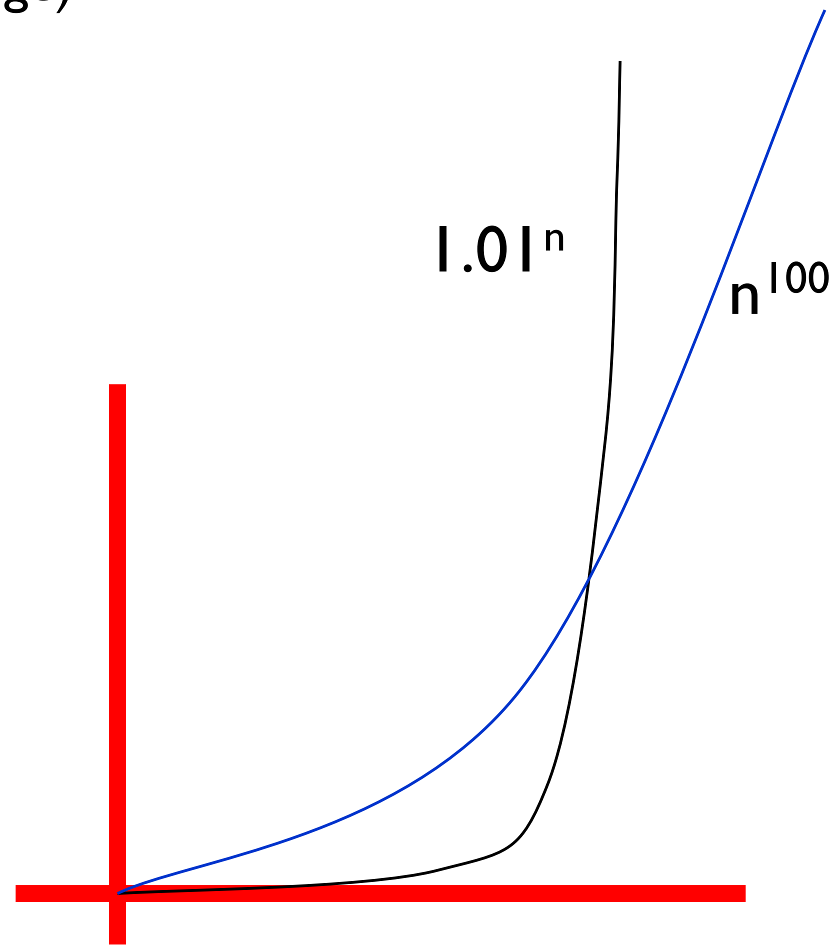
If $f = \Theta(h)$ and $g = \Theta(h)$ then $f + g = \Theta(h)$.

Useful, e.g., for analyzing programs with subroutines.

Proofs are left as exercises.

For all $r > 1$ (no matter how small)
and all $d > 0$, (no matter how large)
$n^d = O(r^n)$

In short, every exponential grows faster than every polynomial!



(To prove this, use calculus tricks like L'Hospital's rule.)

$\boxed{\log_a b = x \text{ means } a^x = b}$

definition

"base"

Examples:

$10^6 = 1{,}000{,}000$, so $\log_{10} 1000000 = 6$

$10^5 = 100{,}000$, so $\log_{10} 100000 = 5$

i.e., $\log_{10}(n) \approx$ number of digits in n; also

$\log_2(n) \approx$ number of bits in n

Key properties:

$\log(x*y) = \log(x) + \log(y)$

$\log(x/y) = \log(x) - \log(y)$

$\log(x^y) = y*\log(x)$

Change-of-base formula:

$\log_a(x) = \log_b(x)/\log_b(a)$

$\log_2(x) = \log_{10}(x)/\log_{10}(2)$

$= \log_{10}(x)/0.30103$

Example: For any a, b>1 $\log_a n$ is $\Theta(\log_b n)$

$$\boxed{\log_a b = x \text{ means } a^x = b}$$ definition

$$a^{\log_a b} = b$$

$$(a^{\log_a b})^{\log_b n} = b^{\log_b n} = n$$

$$\left.\begin{aligned}&(\log_a b)(\log_b n) = \log_a n\\ &c\log_b n = \log_a n \text{ for the constant } c = \log_a b\end{aligned}\right\}$$ change-of-base formula

So:

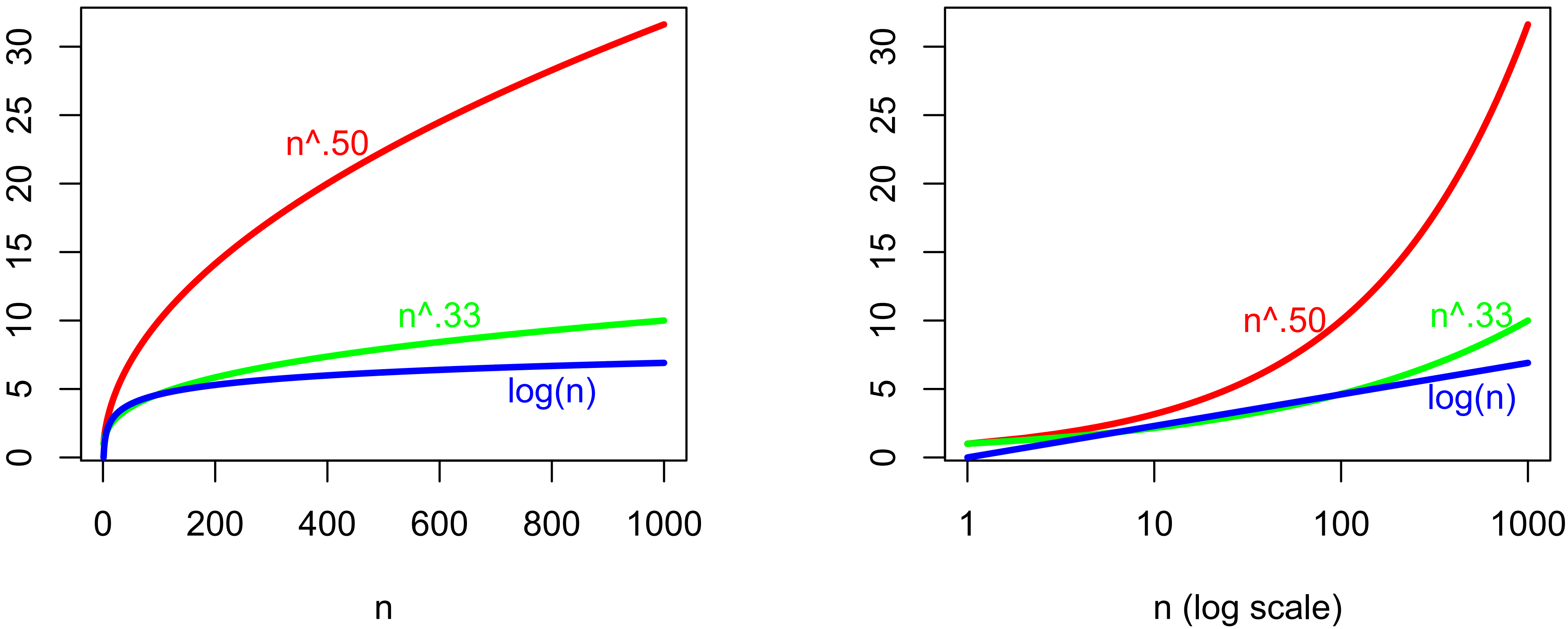
$$\log_b n = \Theta(\log_a n) = \Theta(\log n)$$

Corollary: base of a log *factor* is usually irrelevant, asymptotically. E.g. "O(n log n)" [but $n^{\log_2 8} \neq O(n^{\log_8 8})$]

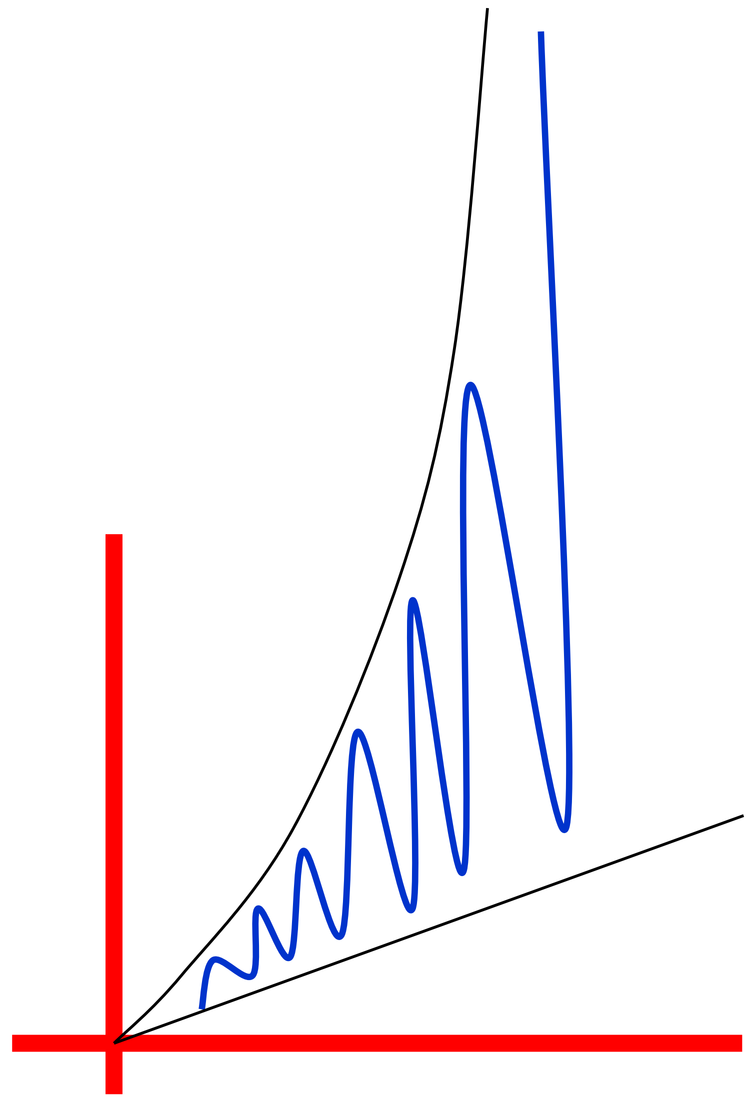
Logarithms:

For all $x > 0$, *(no matter how small)* $\log n = O(n^x)$

log grows slower than every polynomial



$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$



$f(n) \neq \Theta(n^a)$ for any a.

Fortunately, such nasty cases are rare

$n \log n \neq \Theta(n^a)$ for any a, either, but at least it's simpler.

Polynomial Time

P: The set of problems solvable by algorithms with running time $O(n^d)$ for some constant d

(d is a constant independent of the input size n)

Nice scaling property: there is a constant c s.t. *doubling* n, increases time only by a factor of c.

(E.g., $c \sim 2^d$)

Contrast with exponential: For any constant c, there is a d such that $n \to n+d$ increases time by a factor of more than c.

(E.g., c = 100 and d = 7 for 2^n vs 2^{n+7})

polynomial vs exponential growth

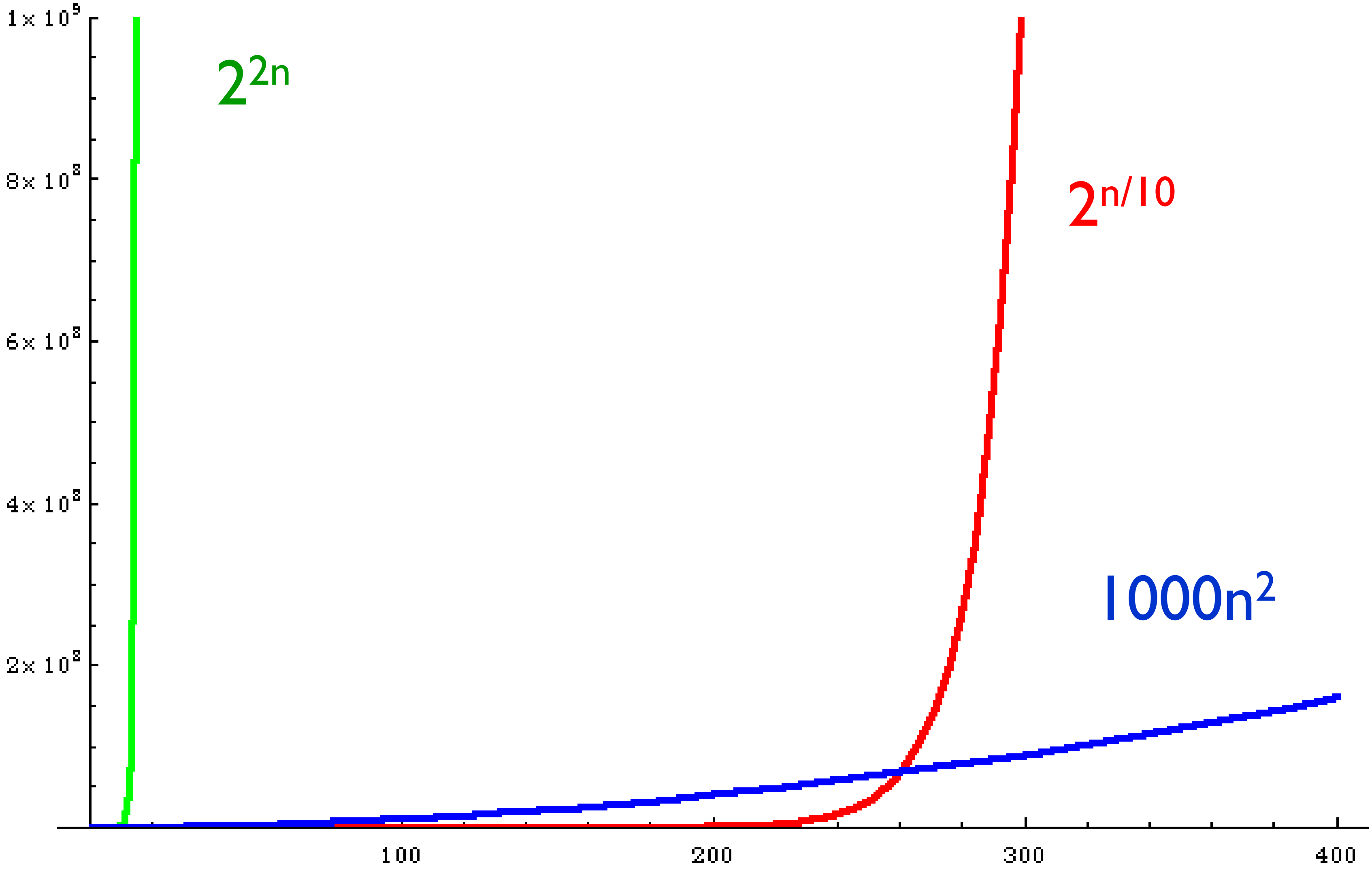


Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

not only get very big, but do so abruptly, which likely yields erratic performance on small instances

Next year's computer will be 2x faster. If I can solve problem of size n_0 today, how large a problem can I solve in the same time next year?

Complexity	Size Increase	E.g. $T=10^{12}$		
$O(n)$	$n_0 \rightarrow 2n_0$	10^{12}	$\rightarrow$	2×10^{12}
$O(n^2)$	$n_0 \rightarrow \sqrt{2}\, n_0$	10^6	$\rightarrow$	1.4×10^6
$O(n^3)$	$n_0 \rightarrow \sqrt[3]{2}\, n_0$	10^4	$\rightarrow$	1.25×10^4
$2^{n/10}$	$n_0 \rightarrow n_0+10$	400	$\rightarrow$	410
2^n	$n_0 \rightarrow n_0+1$	40	$\rightarrow$	41

Point is not that n^{2000} is a nice time bound, or that the differences among n and 2n and n^2 are negligible.

Rather, simple theoretical tools may not easily capture such differences, whereas exponentials are qualitatively different from polynomials, so more amenable to theoretical analysis.

- "My problem is in P" is a starting point for a more detailed analysis
- "My problem is *not* in P" may suggest that you need to shift to a more tractable variant, or otherwise readjust expectations

Summary

A typical initial goal for algorithm analysis is to find a

- reasonably tight, ← i.e., Θ if possible
- asymptotic, ← i.e., O or Θ
- bound on ← usually upper bound
- worst case running time
- as a function of problem size

This is rarely the last word, but often helps separate good algorithms from blatantly poor ones – so you can concentrate on the good ones!

As one important example, poly time algorithms are almost always preferable to exponential time ones.