Homework 0: Solutions

Due Date: This assignment will not be graded! You do not have to submit anything. The solutions are already online for you to compare your answers to, but we strongly recommend you don't look at them until after you attempt the problems.
Collaboration: Since this assignment isn't graded, you're free to discuss these problems with anyone in full detail. But, please read the collaboration policy in the syllabus so you know what will be allowed starting with homework 1.

The following problems have two goals:

(a) To review content you learned in CSE 373 (or equivalent course) but might have forgotten. We have specifically chosen problems for topics that show up early and frequently in the course.

(b) To give you a chance to practice implications before we start using them in more complicated proofs.

1. $\mathcal{O}, \Omega,$ and Θ, oh my!

You've analyzed some code, and found that it runs in $f(n) = 5n^2 + 7n + 3$ time.

For each of the following, state whether it is "true" or "false."
You may want to review asymptotic analysis.

(a) $f(n)$ is $\mathcal{O}\left(n^2\right)$ **Solution:**

True.

(b) $f(n)$ is $\Omega(n^3)$ **Solution:**

False.

(c) $f(n)$ is $\mathcal{O}\left(n^3\right)$ **Solution:**

True. Remember big-$\mathcal{O}$ bounds don't have to be tight.

(d) $f(n)$ is $\mathcal{O}\left([\log(n)]^{100}\right)$ **Solution:**

False. $\log(n)$ to any power is less than n^c for any positive constant c. If you graphed this, you might have been mislead! The crossover point is past 10^{170}, but it does happen!

(e) $f(n)$ is $\Theta(3n^2)$ **Solution:**

True! Constant factors don't matter to big-$\mathcal{O}$ or big-Θ. We'd usually (and you should) write $\Theta(n^2)$ not $\Theta(3n^2)$ but they mean the same thing.

2. Recurrence

$$T(n) = \begin{cases} 3T\left(n/3\right) + 6n^2 & \text{if n} \geq 9 \\ 2 & \text{otherwise} \end{cases}$$

You may want to review here or here. And you might find the math resources to have useful formulas.

(a) Use the tree method or unrolling to caclulate the exact closed form of $T(n)$. You may assume n is a power of 3.
If you use unrolling, show at least three applications of $T()$, and a general form of what i levels of unrolling would give.
If you use the tree method, use the step breakdown in the linked materials to organize your work. [11 points]

Solution:

Tree Method:

- Input size at level i: $n/3^i$
- Each node at level i does $6\left(\frac{n}{3^i}\right)^2 = 6\frac{n^2}{9^i}$.
- At level i we have 3^i nodes.
- Total work at level i: $6\frac{n^2}{9^i} \cdot 3^i = 6\frac{n^2}{3^i}$.
- Last level when $n/3^i < 9$ so when $n/3^i = 3$ (since n is a power of 3) so when $i = \log_3(n) - 1$.
- Work at base case level is: $2 \cdot 3^{\log_3(n)-1} = \frac{2n}{3}$
- Total work:

$$\begin{aligned}
T(n) &= \sum_{i=0}^{\log_3(n)-2} 6\frac{n^2}{3^i} + \frac{2n}{3} \\
&= 6n^2 \sum_{i=0}^{\log_3(n)-2} \left(\frac{1}{3}\right)^i + \frac{2n}{3} \\
&= 6n^2 \frac{(1/3)^{\log_3(n)-1} - 1}{(1/3) - 1} + \frac{2n}{3} \\
&= 6n^2 \frac{3^{\log_3(1/n)+1} - 1}{-2/3} + \frac{2n}{3} \\
&= 6n^2 \frac{3/n - 1}{-2/3} + \frac{2n}{3} \\
&= -9n^2\left(3/n - 1\right) + \frac{2n}{3} \\
&= 9n^2 - 27n + \frac{2n}{3} \\
&= 9n^2 - \frac{79n}{3}
\end{aligned}$$

Solution:

$$
\begin{aligned}
T(n) &= 3T\left(\frac{n}{3}\right) + 6n^2 \\
&= 3\left[3T(\left(\frac{n}{3^2}\right) + 6\left(\frac{n}{3}\right)^2\right] + 6n^2 \\
&= 3^2 T\left(\frac{n}{3^2}\right) + 6\frac{n^2}{3} + 6n^2 \\
&= 3^2\left[3T(\left(\frac{n}{3^3}\right) + 6\left(\frac{n}{3^2}\right)^2\right] + 6\frac{n^2}{3} + 6n^2 \\
&= 3^3 T\left(\frac{n}{3^3}\right) + 6\frac{n^2}{3^2} + 6\frac{n^2}{3} + 6n^2 \\
&\ldots \\
&= 3^i T\left(\frac{n}{3^i}\right) + \sum_{j=0}^{i-1} \frac{6n^2}{3^j} \\
&= 3^{\log_3(n)-1} \cdot 2 + 6n^2 \sum_{j=0}^{\log_3(n)-2} \left(\frac{1}{3}\right)^j \\
&= \frac{2n}{3} + 6n^2\left(\frac{3n-9}{2n}\right) \\
&= 9n^2 - \frac{79n}{3}
\end{aligned}
$$

(b) Check that you got the right leading term by using the Master Theorem. Show the calculation to decide which case of the theorem you are in. [2 points] **Solution:**

$\log_3(3) = 1 < 2$, so we're in the n^c case, and get $\Theta(n^2)$, which matches the leading term of our work!

3. Contrapositive

An "implication" is a statement of the form "if p then q." The "contrapositive" of an implication reverses the order of the implication, and negates both p and q ("if not q, then not p").

For example, the contrapositive of "if it is raining, then I have my umbrella." is "if I do not have my umbrella, then it is not raining."

The "converse" of an implication just switches the order ("if q then p"). For example, the converse of "if it is raining, then I have my umbrella." is "If I have my umbrella, then it is raining."

The contrapositive of an implication and the original implication are equivalent (one is true if and only if the other is). The same is not true of the converse (they might or might not have the same value).

Consider the implication "If G has no negative edge weights, then Dijkstra's algorithm returns the shortest paths of G from the chosen source vertex."

(a) State the contrapositive of the sentence above. [2 points] **Solution:**

"If Dijkstra's returns something other than the shortest paths of G from the source, then G must have at least one negative edge weight."

(b) State the converse of the sentence above. [2 points] **Solution:**

"If Dijkstra's algorithm returns the shortest paths of G from the chosen source vertex, then G has no negative edge weights"

(c) Give an example of a graph G and a chosen source vertex where the converse is false (and thus different from the original claim, which is true). [3 points] **Solution:**

A graph with two vertices (u, v) with only one edge from u to v of weight -1 with u as the source is an example (Dijkstra's will correctly calculate the shortest paths because there is only one vertex to process and so it still goes in the correct order).