Homework 6: Flow and NP-Completeness

Version 2: We clarified the "value" we were looking for in problem 2, and added some extra descriptions of expected length of explanations to parts of problems 4 and 5 that didn't have them already.

Due Date: This assignment is due at 11:59 PM Friday March 5 (Seattle time, i.e. GMT-8).

You will submit the written problems as a PDF to gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit).

Remember to assign pages in your submission!

Collaboration: Please read the full collaboration policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

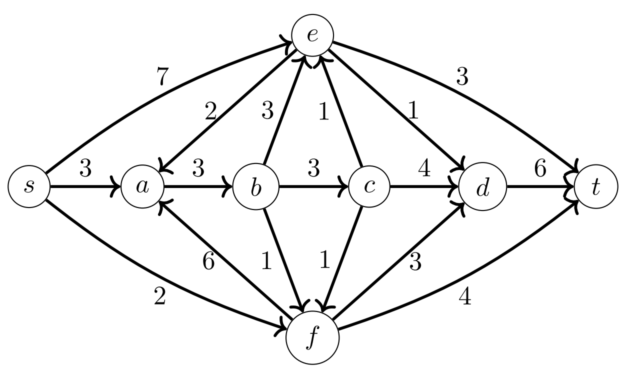
Style and Assumptions: You should check the Style Guide for general principles on how to format your answers. You may use any algorithm described in this list without further explanation. If you wish to use an algorithm you learned in 373 (or 142/143) that isn't on the list, please ask on Ed so we can add it to the list for everyone.

1. HW5 programming

Remember to finish the programming problem from homework 5.

2. Mechanical [10 points]

You are given the following graph where s is the source vertex (where the water starts), t is the sink vertex (where the water ends), and the capacities are written on each edge.



(a) Perform the Ford-Fulkerson algorithm on the graph. Draw the residual graph after each augmenting step (i.e. after each iteration of the while loop from class). And draw the final flow [7 points]

(b) What is the min s, t cut? List both the two sets of vertices and the capacity of the cut (i.e. the value of the cut). [3 points]

3. Assignment [15 points]

New Senate Majority Leader Chuck Schumer needs your help.[1] He has to assign his n senators to the m committees they will serve on, subject to the following conditions:

- Each committee needs **exactly** c democratic senators.
- Each committee needs **at least** one "youthful" democratic senator and one "experienced" democtraic senator.
- Senators will report to him which of the committees they find acceptable. You must not assign a senator to any other committee.
- No senator may serve on more than 5 committees (fewer than 5, down to 0, is acceptable).

You get a list like:
Senator 0, who is youthful, finds committees 0,1 acceptable.
Senator 1, who is youthful, finds committee 0 acceptable.
Senator 2, who is experienced, finds committee 0 acceptable.
Senator 3, who is experienced, finds committees 0,1 acceptable.

For $c = 2$, you could report (for example, multiple assignments are possible):
"Assign senators 0,2 to committee 0"
"Assign senators 0,3 to committee 1"

For $c = 3$ you should report:
"No valid assignment is possible."

(a) Describe a graph you could use a max-flow computation. Remember to give capacities and directions for edges, and choose your source and target! [7 points]

(b) How many vertices does your graph have? In terms of $n, m,$ and/or c (whichever of those you need) [1 point]

(c) Describe how you will interpret a flow in the graph as an answer to the assignment problem. Be sure to mention both finding an assignment and handling there not beign an assignment possible. [3 points]

(d) Describe (in 1-2 sentences each) why any assignment you return must meet all 4 bullets about. [4 points]

[1] or, if you prefer, you may assist new Senate Minority Leader Mitch McConnell with his republican senators

4. Reduction [15 points]

On Homework 5, you had a problem of assigning labels $0, 1, 2$ to employees at a company based on their direct reporting (tree) structure.

Your boss was so impressed with the algorithm you wrote, they decided they want to sell your code to other companies for planning their vacations. There's only one catch, other companies have more complicated (not necessarily tree) reporting structures. Some people might have more than one supervisor, for example. But that should be an easy fix — you can handle that right?

The problem is now the following:

Input: A number n of employees (they will be numbered $0, ..., n-1$).
A list of pairs (i, j) to indicate employee i and j cannot be on vacation the same week (you could get **any** list of pairs, not just a list representing a tree).

Output: The minimum number of 0 labels that are required in a labeling that has no listed pairs with the same label, or ∞ if it's not possible to label.

For example, on input
$n = 3$
$(0, 1), (1, 2), (0, 2)$
The correct output is 1

on input
$n = 4$
$(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)$
The correct output is ∞.

(a) Give a reduction that shows the problem is now NP-hard.
Remember you reduce FROM the known NP-complete problem (this one always feels backward to Robbie, be careful!) [8 points]

(b) Explain why your reduction is correct. Remember to handle both directions!
We have 1-2 sentences per direction. [4 points]

(c) Did we prove P=NP on homework 5? Explain why or why not. (Our explanation is 2 sentences) [3 points]

5. Reduction

A **Burr Path**[2] is a path that visits exactly $1/3$ of the vertices in a graph.

The **Burr Path Problem** is the following:
Input: A directed, unweighted graph
Output:
`True` if there is a path in the graph that visits exactly $1/3$ of the vertices
`False` otherwise

For example:
On a 5 vertex graph, the answer is always `False` (there is no such thing as a path that uses $\frac{5}{3}$ vertices)
On a 9 vertex cycle, the answer is `True`, any three consectuive vertices (and the edges between them) form a Burr Path.

Burr Path is NP-complete.

Someone suggests the following reduction from Hamiltonian Path:
"Don't change the graph at all, just take the output of the hypothetical Burr Path solver. If there is a Hamiltonian Path, there must be a Burr Path too."

The reduction is not valid.

(a) Give a graph on which the reduction fails, and explain why it is incorrect (our explanation is 2 sentences). [5 points]

(b) Give a correct reduction from Hamiltonian Path to Burr Path. [9 points]

(c) Explain why your reduction is correct. Remember to handle both directions! Our explanations are 2-3 sentences each. [6 points]

[2] Like a Hamiltonian Path, but not as good. Hamiltonian Paths aren't named after Alexander Hamilton, but this is the best name we could think of.