CSE 417
Algorithms and Complexity

Richard Anderson - Lecture 27
Coping with NP-Completeness and Beyond

Announcements
• No final exam
• Homework 9 – Due March 13, 5:00 pm
• Homework 10 – Due March 18, 5:00 pm
• NP-Completeness
• Counts as a regular HW
• Office hours by zoom

NP Completeness:
The story so far

There are a whole bunch of other important problems which are NP-Complete

Populating the NP-Completeness Universe
• Circuit Sat \leq_p 3-SAT
• 3-SAT \leq_p Independent Set
• 3-SAT \leq_p Vertex Cover
• Independent Set \leq_p Clique
• 3-SAT \leq_p Hamiltonian Circuit
• Hamiltonian Circuit \leq_p Traveling Salesman
• 3-SAT \leq_p Integer Linear Programming
• 3-SAT \leq_p Graph Coloring
• 3-SAT \leq_p Subset Sum
• Subset Sum \leq_p Scheduling with Release times and deadlines

Satisfiability
Literal: A Boolean variable or its negation.

Clause: A disjunction of literals.

Conjunctive normal form: A propositional formula \( \Phi \) that is the conjunction of clauses.

SAT: Given CNF formula \( \Phi \), does it have a satisfying truth assignment?

3-SAT: SAT where each clause contains exactly 3 literals.

Ex: \((x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (x_2 \lor x_3) \land (\overline{x_1} \lor x_2 \lor \overline{x_3})\)

Yes: \(x_1 = \text{true}, x_2 = \text{false}, x_3 = \text{false}\)

Matching
Two dimensional matching

Three dimensional matching (3DM)
**3-SAT \( \leq_p \) 3DM**

Truth Setting Gadget

- **Truth Setting Gadget**
- **Clause gadget for \( X \lor Y \lor Z \)**
- **Garbage Collection Gadget** (Many copies)

**Exact Cover (sets of size 3) XC3**

Given a collection of sets of size 3 of a domain of size 3N, is there a sub-collection of N sets that cover the sets:

\[
\begin{align*}
& (A, B, C), (D, E, F), (A, B, G), \\
& (A, C, I), (B, E, G), (A, G, I), \\
& (B, D, F), (C, E, I), (C, D, H), \\
& (D, G, I), (D, F, H), (E, H, I), \\
& (F, G, H), (F, H, I)
\end{align*}
\]

3DM \( \leq_p \) XC3

**Graph Coloring**

- **NP-Complete**
  - Graph K-coloring
  - Graph 3-coloring
- **Polynomial**
  - Graph 2-coloring

**Number Problems**

- **Subset sum problem**
  - Given natural numbers \( w_1, \ldots, w_n \) and a target number \( W \), is there a subset that adds up to exactly \( W \)?
- **Subset sum problem is NP-Complete**
- **Subset Sum problem can be solved in \( O(nW) \) time**

**3-SAT \( \leq_p \) 3 Colorability**

Truth Setting Gadget

- **Truth Setting Gadget**
- **Clause Testing Gadget** (Can be colored if at least one input is T)
**XC3 \( \leq_p \) SUBSET SUM**

Idea: Represent each set as a large integer, where the element \( x_i \) is encoded as \( D^i \) where \( D \) is an integer

\[ \{x_3, x_5, x_9\} \Rightarrow D^3 + D^5 + D^9 \]

Does there exist a subset that sums to exactly \( D^1 + D^2 + D^3 + \ldots + D^{n-1} + D^n \)?

**Detail:** How large is \( D \)? We need to make sure that we do not have any carries, so we can choose \( D = m+1 \), where \( m \) is the number of sets.

**Integer Linear Programming**

- Linear Programming – maximize a linear function subject to linear constraints
- Integer Linear Programming – require an integer solution
- NP Completeness reduction from 3-SAT

Use 0-1 variables for \( x_i \)'s

Constraint for clause \( x_1 \lor x_2 \lor x_3 \)

\[ x_i + (1 - x_i) + (1 - x_j) > 0 \]

**Coping with NP-Completeness**

- Approximation Algorithms
- Exact solution via Branch and Bound
- Local Search

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I can't find an efficient algorithm, but neither can all these famous people.
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**Multiprocessor Scheduling**

- Unit execution tasks
- Precedence graph
- K-Processors

- Polynomial time for \( k=2 \)
- Open for \( k = \text{constant} \)
- NP-complete is \( k \) is part of the problem

**Highest level first is 2-Optimal**

Choose \( k \) items on the highest level

Claim: number of rounds is at least twice the optimal.

**Christofides TSP Algorithm**

- Undirected graph satisfying triangle inequality

1. Find MST
2. Add additional edges so that all vertices have even degree
3. Build Eularian Tour

```
3/2 Approximation
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```
1  2  4
6

2  3  4  5

3  2

4

5

6

```

```
3/2 Approximation
```
Christofides Algorithm

Bin Packing

- Given N items with weight $w_i$, pack the items into as few unit capacity bins as possible
- Example: .3, .3, .3, .3, .4, .4

First Fit Packing

- First Fit
  - Theorem: $FF(I)$ is at most $17/10 \text{Opt}(I) + 2$
- First Fit Decreasing
  - Theorem: $FFD(I)$ is at most $11/9 \text{Opt}(I) + 4$

Branch and Bound

- Brute force search – tree of all possible solutions
- Branch and bound – compute a lower bound on all possible extensions
  - Prune sub-trees that cannot be better than optimal

Branch and Bound for TSP

- Enumerate all possible paths
- Lower bound: Current path cost plus MST of remaining points
- Euclidean TSP
  - Points on the plane with Euclidean Distance
  - Sample data set: State Capitals

Local Optimization

- Improve an optimization problem by local improvement
  - Neighborhood structure on solutions
  - Travelling Salesman 2-Opt (or K-Opt)
  - Independent Set Local Replacement
What we don’t know

• P vs. NP

If P $\neq$ NP, is there anything in between

• Yes, Ladner [1975]
• Problems not known to be in P or NP Complete
  – Factorization
  – Discrete Log
  – Graph Isomorphism
  
  Solve $g^x = b$ over a finite group.

Complexity Theory

• Computational requirements to recognize languages
• Models of Computation
• Resources
• Hierarchies

Time complexity

• P: (Deterministic) Polynomial Time
• NP: Non-deterministic Polynomial Time
• EXP: Exponential Time

Space Complexity

• Amount of Space (Exclusive of Input)
• L: Logspace, problems that can be solved in $O(\log n)$ space for input of size $n$
  – Related to Parallel Complexity

• PSPACE, problems that can be required in a polynomial amount of space

So what is beyond NP?

ON BEYOND ZEBRA!

by Dr. Seuss...
NP vs. Co-NP

- Given a Boolean formula, is it true for some choice of inputs
- Given a Boolean formula, is it true for all choices of inputs

Problems beyond NP

- Exact TSP, Given a graph with edge lengths and an integer K, does the minimum tour have length K
- Minimum circuit, Given a circuit C, is it true that there is no smaller circuit that computes the same function as C

Polynomial Hierarchy

- Level 1
  - $\exists X_1 \Phi(X_1), \forall X_1 \Phi(X_1)$
- Level 2
  - $\forall X_1 \exists X_2 \Phi(X_1,X_2), \exists X_1 \forall X_2 \Phi(X_1,X_2)$
- Level 3
  - $\forall X_1 \exists X_2 \forall X_3 \Phi(X_1,X_2,X_3), \exists X_1 \forall X_2 \exists X_3 \Phi(X_1,X_2,X_3)$

Polynomial Space

- Quantified Boolean Expressions
  - $\exists X_1 \forall X_2 \exists X_3 ... \exists X_n \forall X_1 \Phi(X_1,X_2,X_3...X_n,X_1)$
- Space bounded games
  - Competitive Facility Location Problem
  - N x N Chess
- Counting problems
  - The number of Hamiltonian Circuits