CSE 417 Algorithms

Lecture 19, Autumn 2020

Dynamic Programming

Announcements

- Dynamic Programming Reading:
  - 6.1-6.2, Weighted Interval Scheduling
  - 6.3 Segmented Least Squares
  - 6.4 Knapsack and Subset Sum
  - 6.6 String Alignment
    - 6.7* String Alignment in linear space
  - 6.8 Shortest Paths (again)
  - 6.9 Negative cost cycles
    - How to make an infinite amount of money

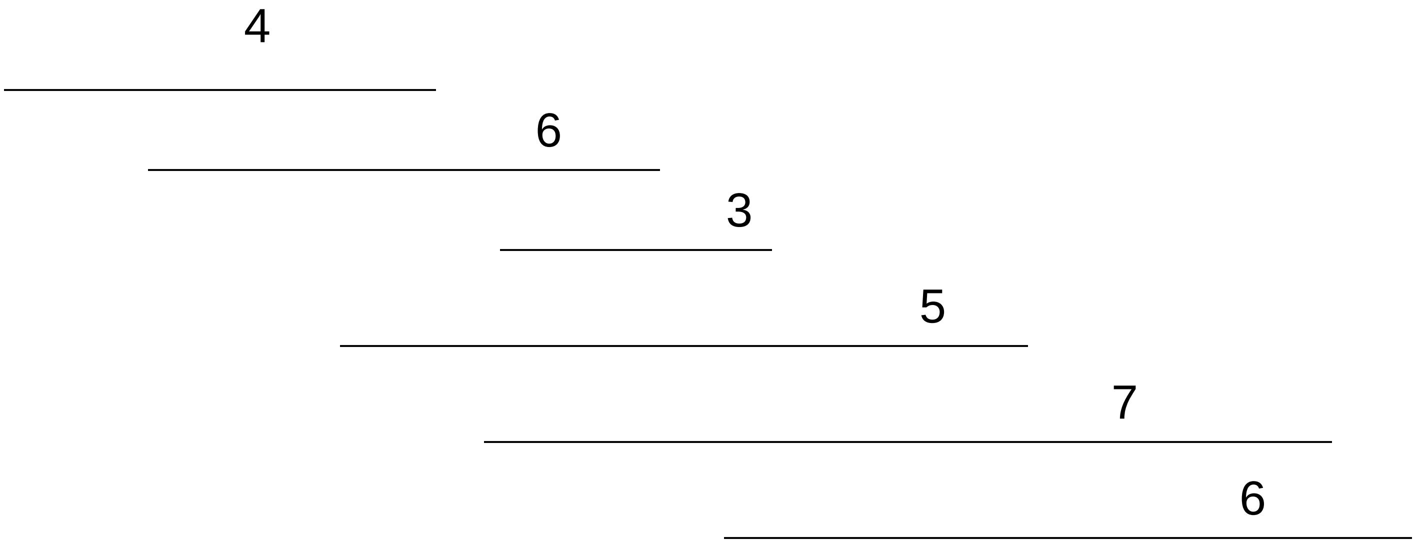
Dynamic Programming

- The most important algorithmic technique covered in CSE 417
- Key ideas
  - Express solution in terms of a polynomial number of sub problems
  - Order sub problems to avoid recomputation

Intervals sorted by end time

Dynamic Programming

- Weighted Interval Scheduling
- Given a collection of intervals $I_1,\ldots,I_n$ with weights $w_1,\ldots,w_n$, choose a maximum weight set of non-overlapping intervals



Intervals sorted by end time

Optimality Condition

- Opt[j] is the maximum weight independent set of intervals $I_1, I_2, \ldots, I_j$
- Opt[j] = max(Opt[j – 1], w_j + Opt[p[j]])
  - Where p[j] is the index of the last interval which finishes before I_j starts

Algorithm

```
MaxValue(j) =
   if j = 0 return 0
   else
        return max( MaxValue(j-1),
                    w_j + MaxValue(p[ j ]))
```

Worst case run time: 2^n

A better algorithm

M[j] initialized to -1 before the first recursive call for all j

```
MaxValue(j) =
    if j = 0 return 0;
    else if M[ j ] != -1 return M[ j ];
    else
        M[ j ] = max(MaxValue(j-1), w_j + MaxValue(p[ j ]));
        return M[ j ];
```

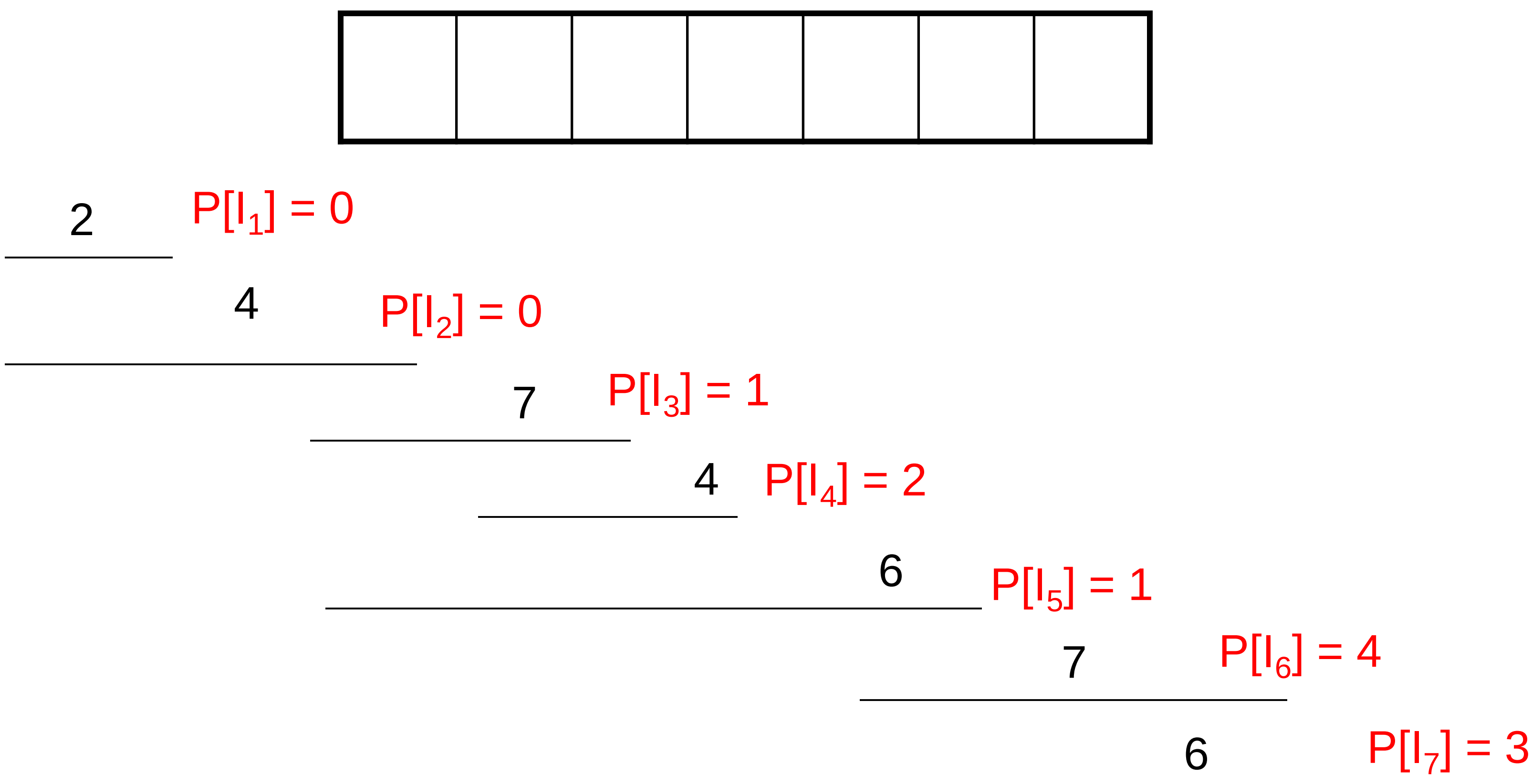
Iterative Algorithm

Express the MaxValue algorithm as an iterative algorithm

```
MaxValue {

}
```

Fill in the array with the Opt values

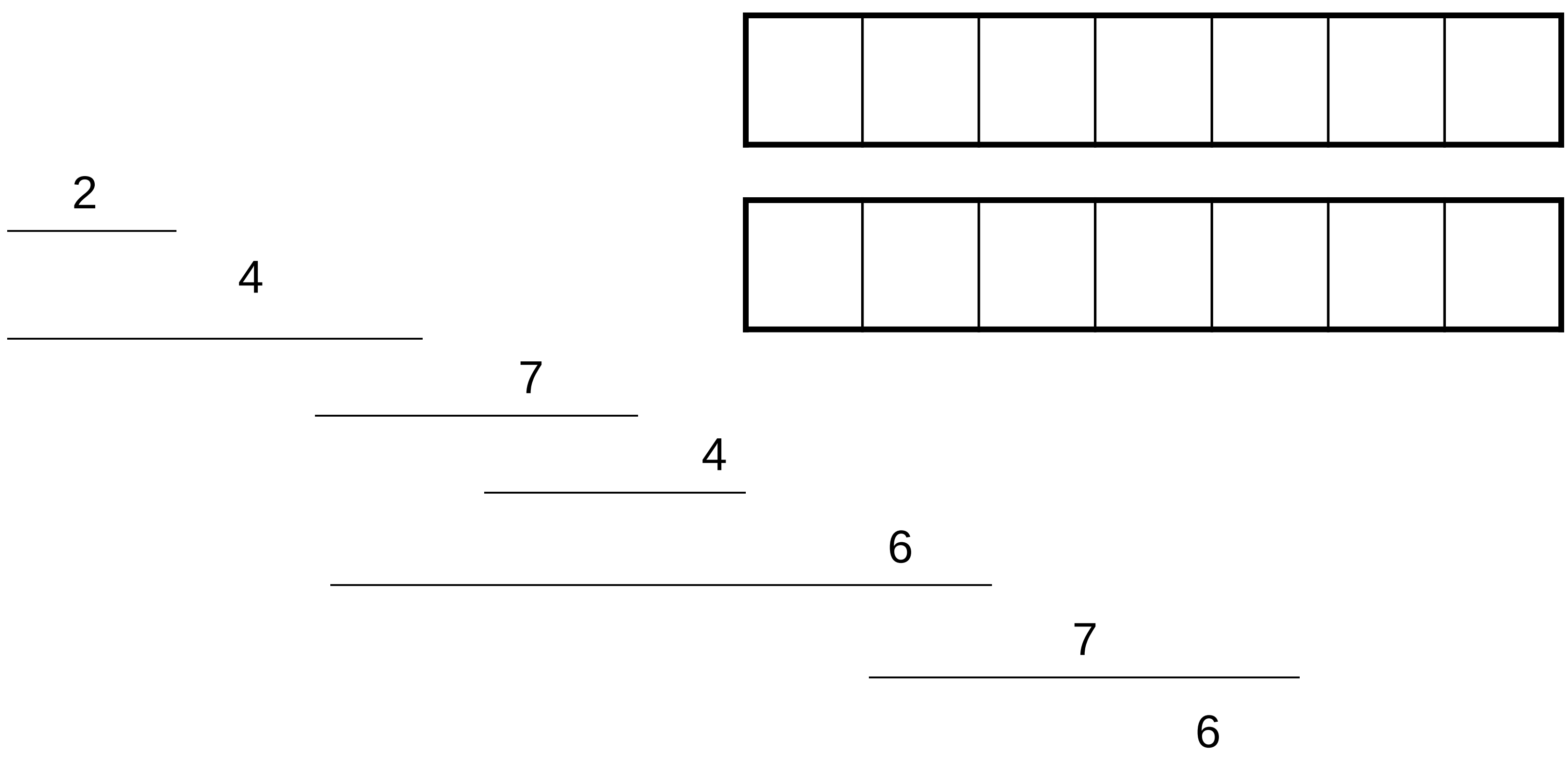
$$Opt[\ j\ ] = \max\ (Opt[\ j - 1],\ w_j + Opt[\ p[\ j\ ]\ ])$$



Computing the solution

$\text{Opt}[\,j\,] = \max(\text{Opt}[\,j - 1], w_j + \text{Opt}[\,p[\,j\,]\,])$

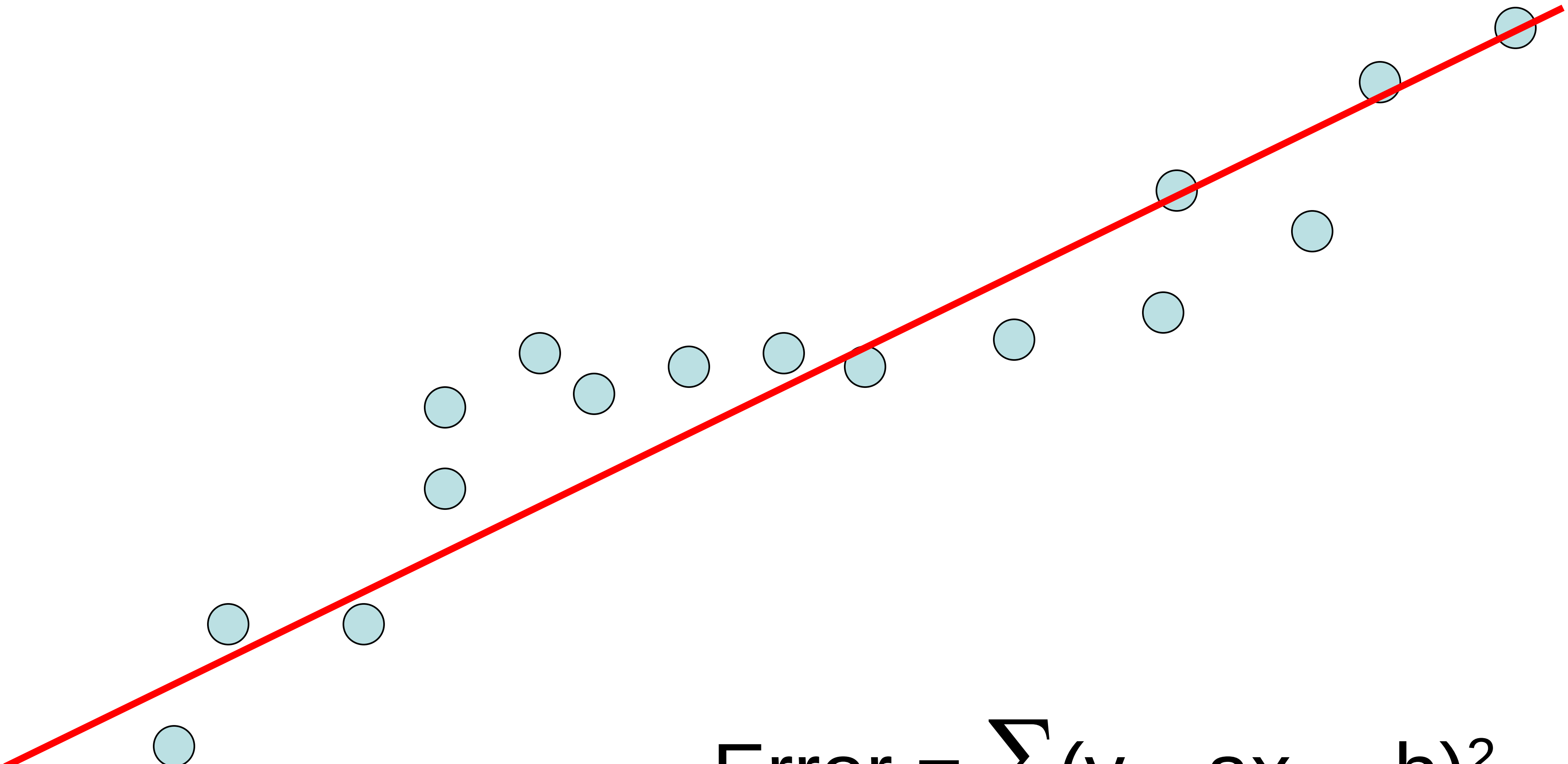
Record which case is used in Opt computation



Iterative Algorithm

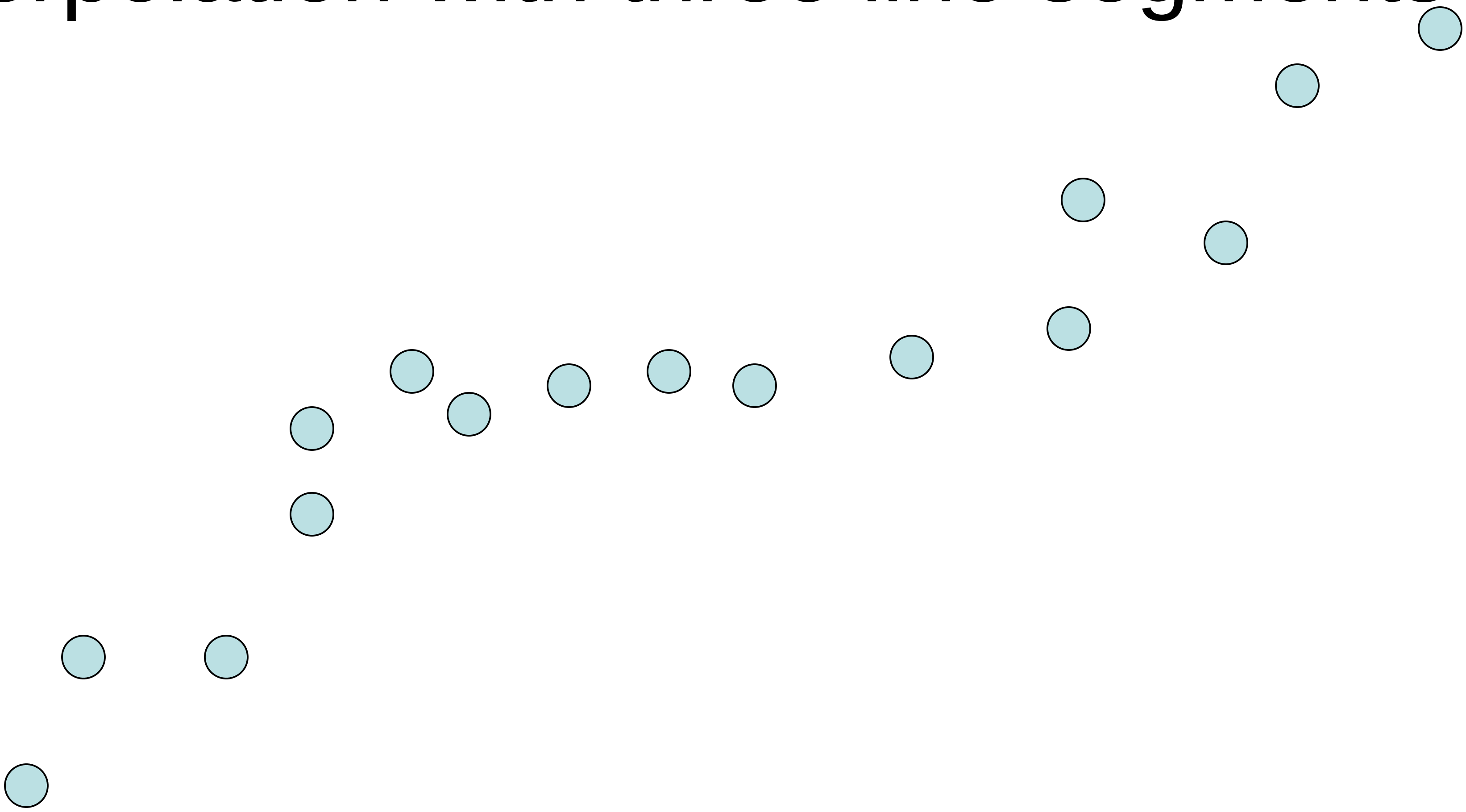
```
int[] M = new int[n+1];
char[] R = new char[n+1];

M[0] = 0;
for (int j = 1; j < n+1; j++){
        v1 = M[j-1];
        v2 = W[j] + M[P[j]];
        if (v1 > v2) {
                M[j] = v1;
                R[j] = ‘A’;
        }
        else {
                M[j] = v2;
                R[j] = ‘B’;
        }
}
```

Optimal linear interpolation

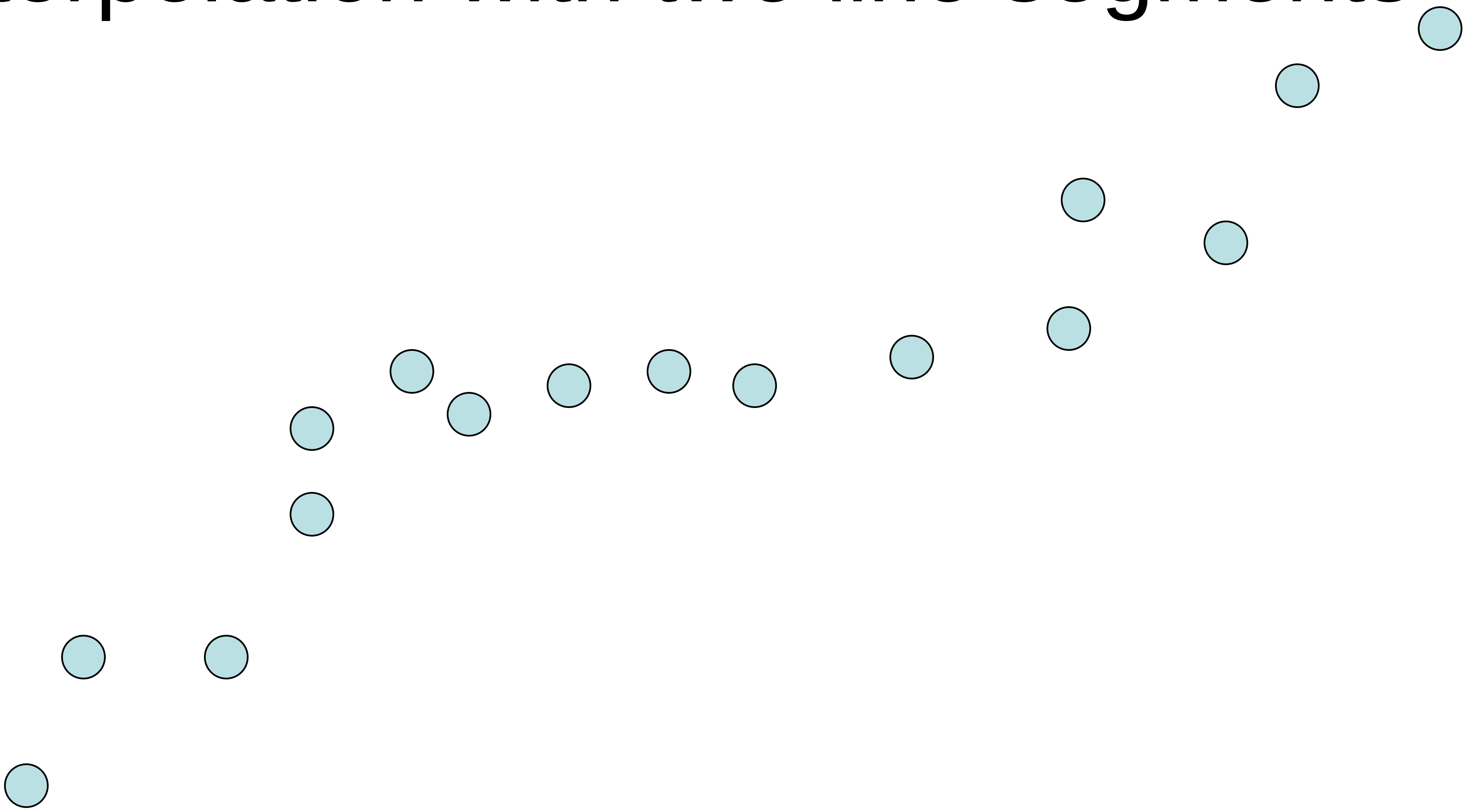


$$\text{Error} = \sum (y_i - ax_i - b)^2$$

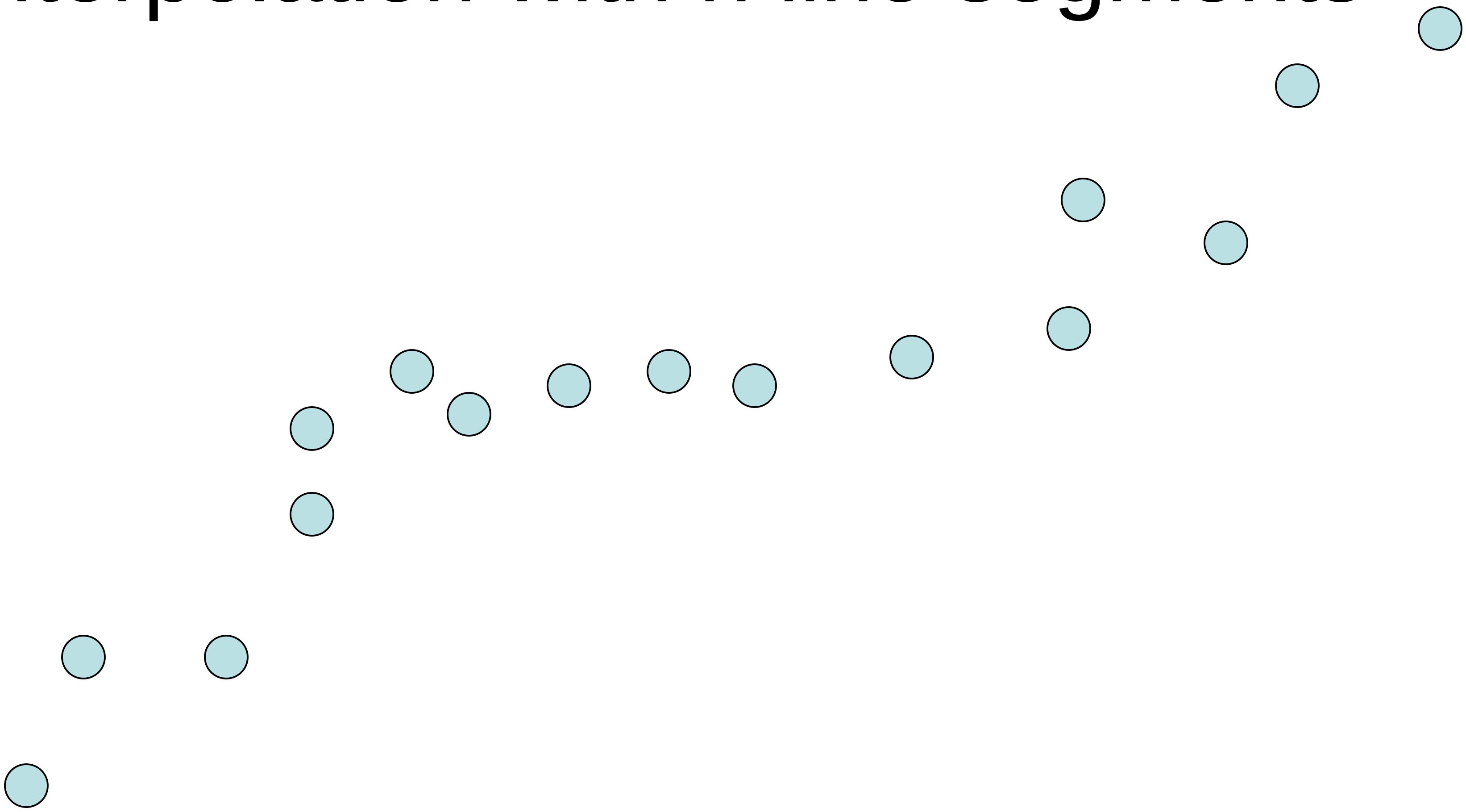
What is the optimal linear interpolation with three line segments



What is the optimal linear interpolation with two line segments

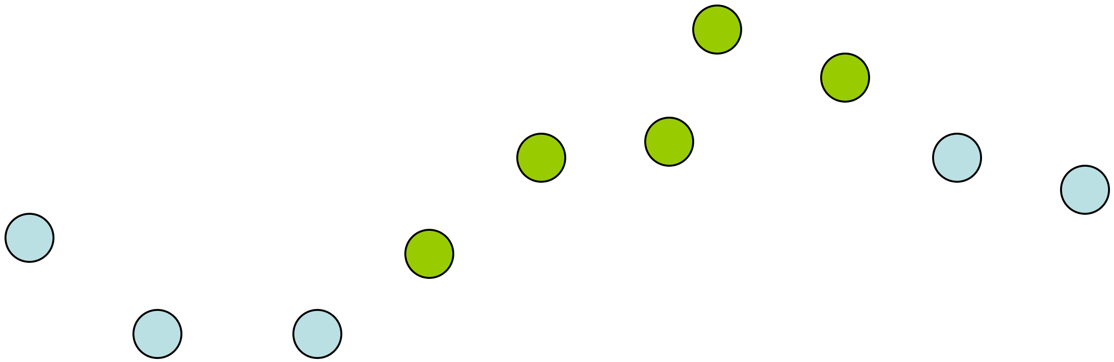


What is the optimal linear interpolation with n line segments



Notation

- Points p_1, p_2, . . ., p_n ordered by x-coordinate ($p_i = (x_i, y_i)$)
- $E_{i,j}$ is the least squares error for the optimal line interpolating p_i, . . . p_j



Optimal interpolation with two segments

- Give an equation for the optimal interpolation of $p_1,\ldots,p_n$ with two line segments

- $E_{i,j}$ is the least squares error for the optimal line interpolating $p_i, \ldots p_j$

Optimal interpolation with k segments

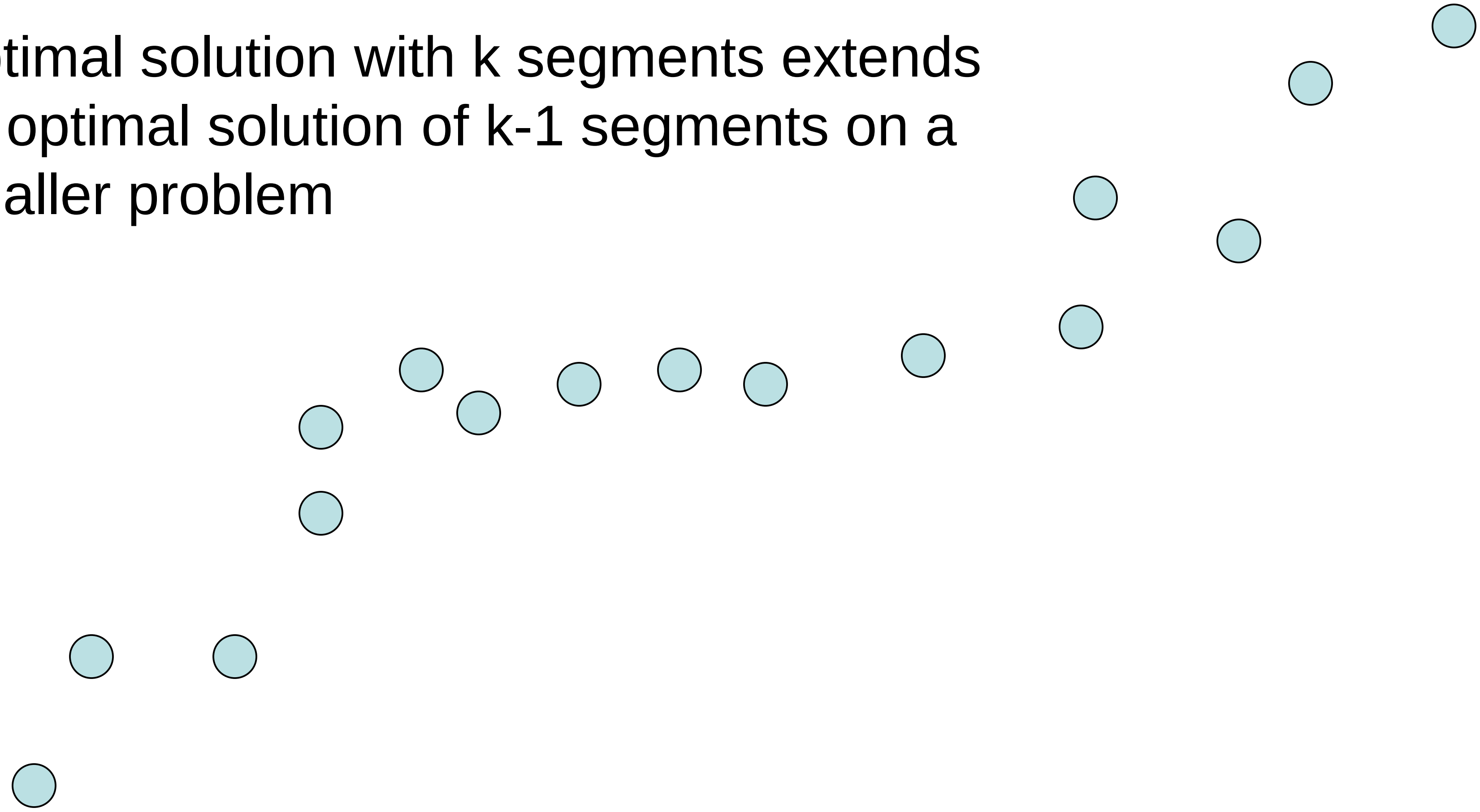
- Optimal segmentation with three segments
  - $\text{Min}_{i,j}\{E_{1,i} + E_{i,j} + E_{j,n}\}$
  - $O(n^2)$ combinations considered
- Generalization to k segments leads to considering $O(n^{k-1})$ combinations

$Opt_k[\ j\ ]$: Minimum error approximating $p_1 \ldots p_j$ with k segments

How do you express $Opt_k[\ j\ ]$ in terms of $Opt_{k-1}[1], \ldots, Opt_{k-1}[\ j\ ]$?

Optimal sub-solution property

Optimal solution with k segments extends an optimal solution of k-1 segments on a smaller problem



Optimal multi-segment interpolation

Compute Opt[k, j] for 0 < k < j < n

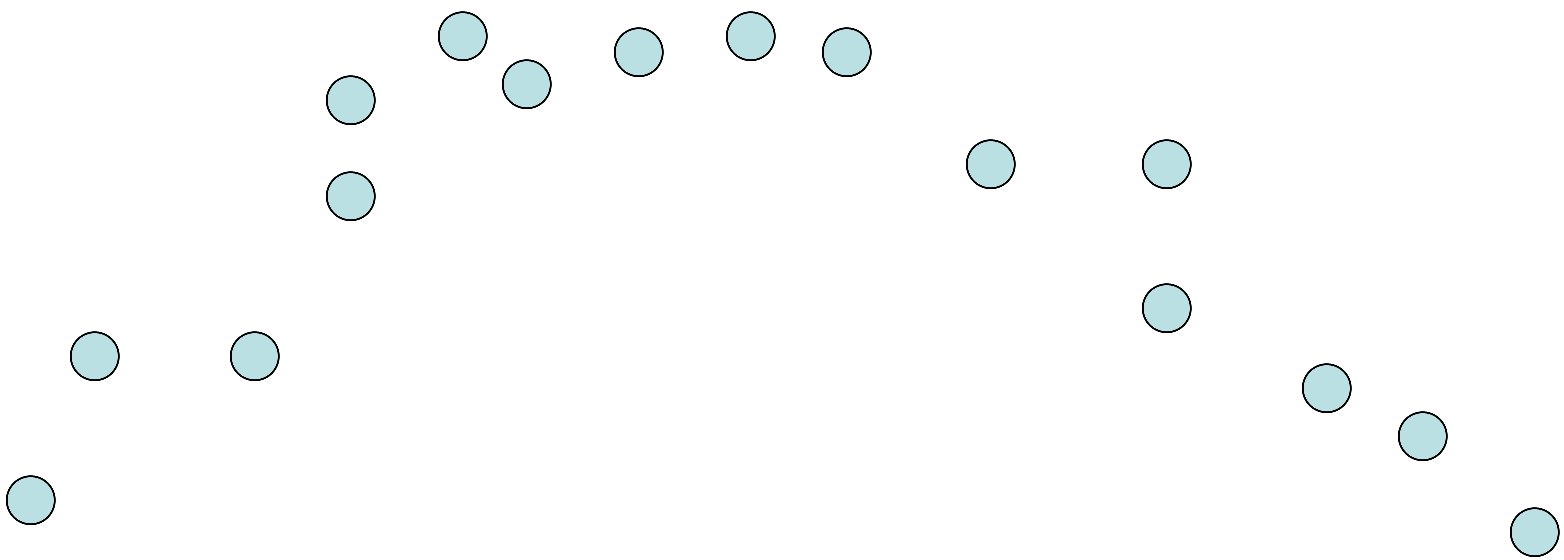
```
for j = 1 to n
    Opt[1, j] = E1,j;

for k = 2 to n-1
    for j = 2 to n
        t = E1,j
        for i = 1 to j-1
            t = min(t, Opt[k-1, i] + Ei,j)
        Opt[k, j] = t
```

Determining the solution

- When Opt[k,j] is computed, record the value of i that minimized the sum
- Store this value in an auxiliary array
- Use to reconstruct solution

Variable number of segments

- Segments not specified in advance
- Penalty function associated with segments
- Cost = Interpolation error + C x #Segments



Penalty cost measure

- Opt[j] = min($E_{1,j}$, $\min_i$(Opt[i] + $E_{i,j}$ + P))