CSE 417
Algorithms and Complexity

Autumn 2020
Lecture 18
Divide and Conquer Algorithms

Announcements

- Homework 6, Due Wednesday, Nov 18
  - No class Wednesday, Nov 11

Divide and Conquer Algorithms

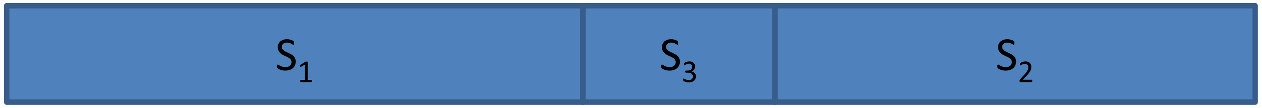
- Mergesort, Quicksort
- Strassen’s Algorithm
- Median
- Inversion counting
- Closest Pair Algorithm (2d)
- Integer Multiplication (Karatsuba’s Algorithm)

Select the k-th largest from an array

- *Selection*, given n numbers and an integer k, find the k-th largest
- Median is a special case
- The standard approach is to use a quicksort like algorithm
  - But with one recursive problem
- The difficulty is ensuring a good split
  - Worst case $O(n^2)$ time

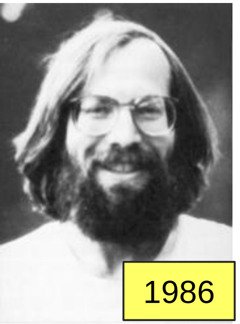
Select(A, k)

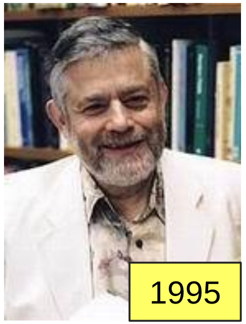
Select(A, k){
    Choose a pivot element x from A
    $S_1 = \{y \text{ in } A \mid y < x\}$
    $S_2 = \{y \text{ in } A \mid y > x\}$
    $S_3 = \{y \text{ in } A \mid y = x\}$
    if ($|S_2| >= k$)
        return Select(S_2, k)
    else if ($|S_2| + |S_3| >= k$)
        return x
    else
        return Select(S_1, $k - |S_2| - |S_3|$)
}

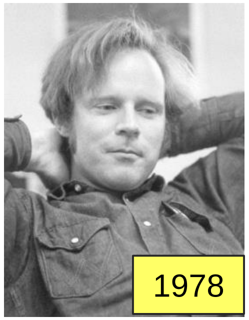


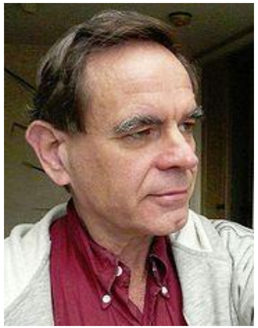
What to know about median finding

- The key to the algorithm is pivot selection
- Choosing a random pivot works well
- Improved random pivot selection: median of three
- Randomized algorithms can find median with 3/2 n comparisons
- Deterministic median finding is harder
  - BFPRT Algorithm guarantees a 3n/4-n/4 split



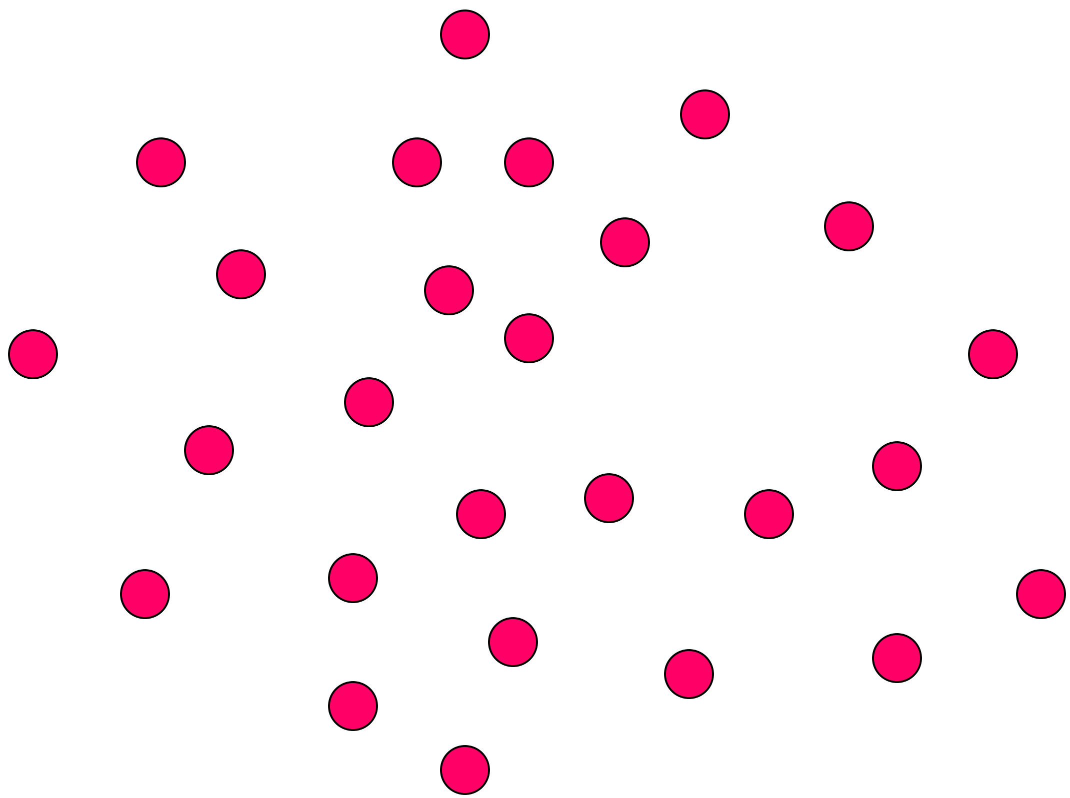






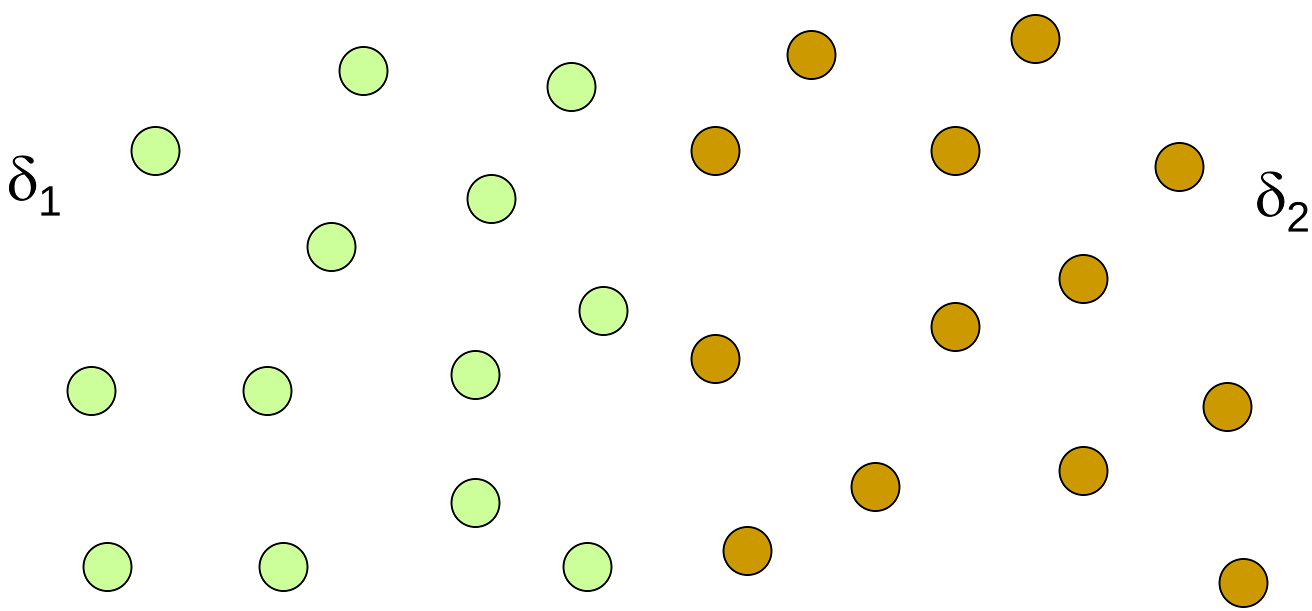

Closest Pair Problem (2D)

- Given a set of points find the pair of points p, q that minimizes dist(p, q)



Divide and conquer

- If we solve the problem on two subsets, does it help? (Separate by median x coordinate)



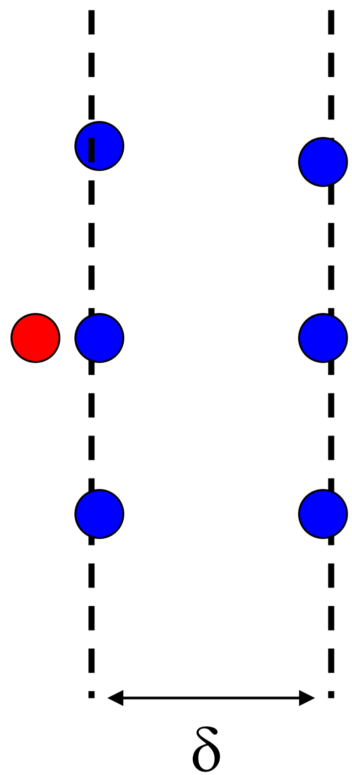
Packing Lemma

Suppose that the minimum distance between points is at least δ, what is the maximum number of points that can be packed in a ball of radius δ?

Combining Solutions

- Suppose the minimum separation from the sub problems is δ
- In looking for cross set closest pairs, we only need to consider points with δ of the boundary
- How many cross border interactions do we need to test?

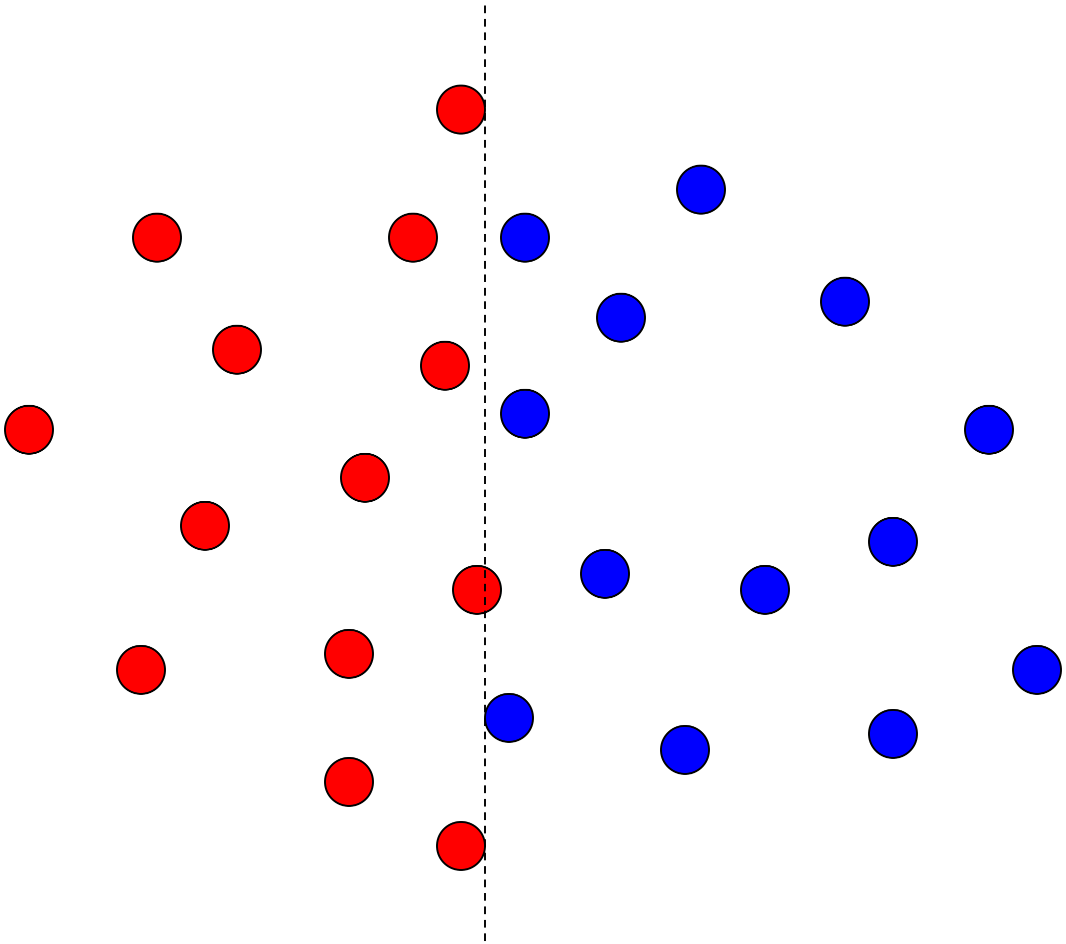
A packing lemma bounds the number of distances to check



Details

- Preprocessing: sort points by y
- Merge step
  - Select points in boundary zone
  - For each point in the boundary
    - Find highest point on the other side that is at most δ above
    - Find lowest point on the other side that is at most δ below
    - Compare with the points in this interval (there are at most 6)

Identify the pairs of points that are compared in the merge step following the recursive calls



Algorithm run time

- After preprocessing:
  - $T(n) = cn + 2\,T(n/2)$

Integer Arithmetic

$$
\begin{array}{rr}
 & 9715480283945084383094856701043643845790217965702956767 \\
+ & 1242431098234099057329075097179898430928779579277597977 \\
\hline
\end{array}
$$

Runtime for standard algorithm to add two n digit numbers:

$$
\begin{array}{rr}
 & 2095067093034680994318596846868779409766717133476767930 \\
\times & 5920175091777634709677679342929097012308956679993010921 \\
\hline
\end{array}
$$

Runtime for standard algorithm to multiply two n digit numbers:

Recursive Multiplication Algorithm (First attempt)

$x = x_1 2^{n/2} + x_0$

$y = y_1 2^{n/2} + y_0$

$xy = (x_1 2^{n/2} + x_0)(y_1 2^{n/2} + y_0)$

$\quad = x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0$

Recurrence:

Run time:

Simple algebra

$x = x_1 2^{n/2} + x_0$

$y = y_1 2^{n/2} + y_0$

$xy = x_1y_1 2^n + (x_1y_0 + x_0y_1) 2^{n/2} + x_0y_0$

$p = (x_1 + x_0)(y_1 + y_0) = x_1y_1 + x_1y_0 + x_0y_1 + x_0y_0$

Karatsuba's Algorithm

Multiply n-digit integers x and y

Let $x = x_1 2^{n/2} + x_0$ and $y = y_1 2^{n/2} + y_0$
Recursively compute

$a = x_1 y_1$
$b = x_0 y_0$
$p = (x_1 + x_0)(y_1 + y_0)$

Return $a2^n + (p - a - b)2^{n/2} + b$

Recurrence: $T(n) = 3T(n/2) + cn$

$\log_2 3 = 1.58496250073\ldots$

Fast Integer Multiplication

- Grade School $O(n^2)$
- Karatsuba $O(n^{1.58})$
- Toom-Cook $O(n^{1.46})$ [For 3 pieces]
  - $O(n^{1+eps})$ [For k pieces]
- Schonhage-Strassen
  - Fast Fourier Transform based algorithm
  - $O(n \log n \log\log n)$
  - Becomes practical for ~25,000 digits

No class Wednesday

- Dynamic Programming starting on Friday