CSE 417
Algorithms and Complexity

Autumn 2020
Lecture 16
Divide and Conquer Algorithms

Announcements

- Homework 5, Due Friday

Matrix Multiplication

- N X N Matrix,   A B = C

```
for (int i = 0; i < n; i++)
        for (int j = 0;  j < n; j++) {
                int t = 0;
                for (int k = 0; k < n; k++)
                        t = t + A[i][k] * B[k][j];
                C[i][j] = t;
        }
```

Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = ae + bf$

$s = ag + bh$

$t = ce + df$

$u = cg + dh$

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

Recursive Matrix Multiplication

- How many recursive calls are made at each level?

- How much work in combining the results?

- What is the recurrence?

What is the run time for the recursive Matrix Multiplication Algorithm?

- Recurrence:

Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 – p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b – d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a – c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g – h)$

$p_6 = d(f – e)$

$p_7 = (c + d)e$

From Aho, Hopcroft, Ullman 1974

Recurrence for Strassen's Algorithms

- $T(n) = 7\ T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 = 2.8073549221$

Strassen's Algorithms

- Treat n x n matrices as 2 x 2 matrices of n/2 x n/2 submatrices
- Use Strassen's trick to multiply 2 x 2 matrices with 7 multiplies
- Base case standard multiplication for single entries
- Recurrence: $T(n) = 7\ T(n/2) + cn^2$
- Solution is $O(7^{\log n}) = O(n^{\log 7})$ which is about $O(n^{2.807})$

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Inversion Problem

- Let $a_1, \ldots a_n$ be a permutation of $1 \ldots n$
- (a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

- Problem: given a permutation, count the number of inversions
- This can be done easily in $O(n^2)$ time
  - Can we do better?

Application

- Counting inversions can be use to measure how close ranked preferences are
  - People rank 20 movies, based on their rankings you cluster people who like that same type of movie

Counting Inversions

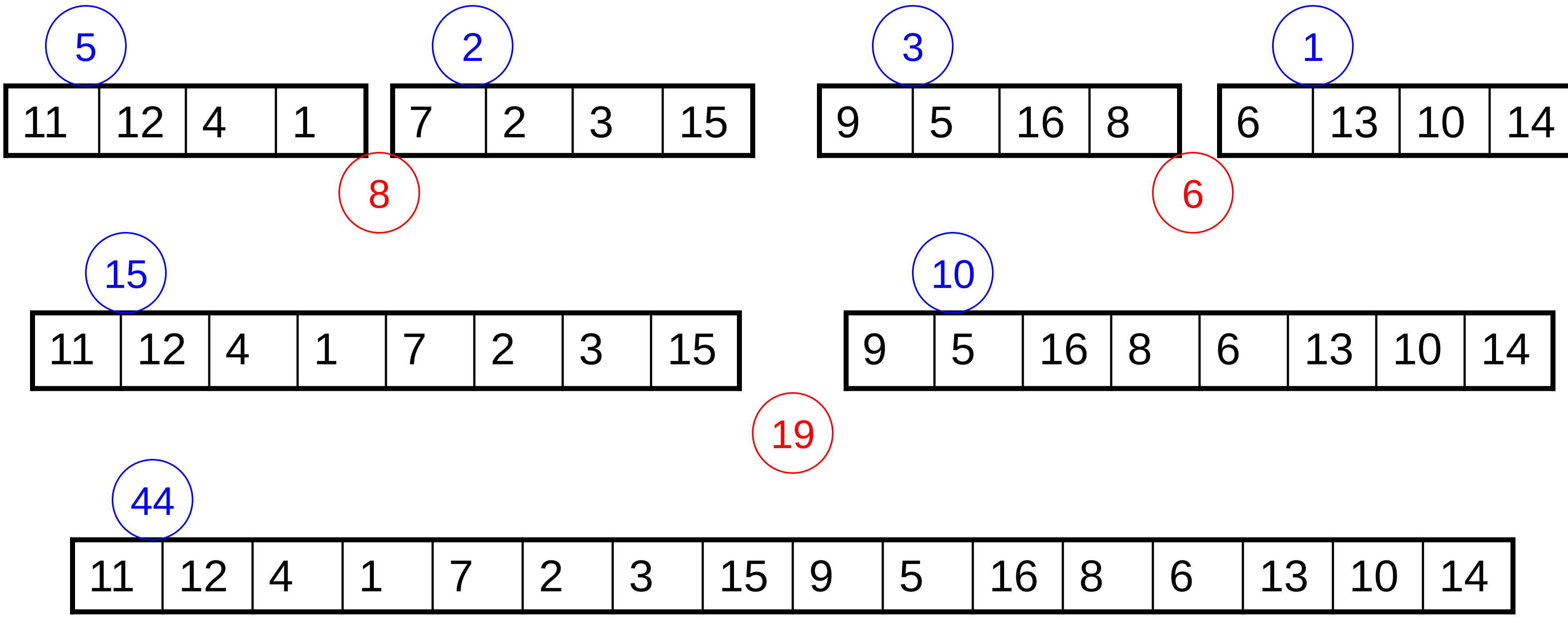
11	12	4	1	7	2	3	15	9	5	16	8	6	13	10	14

Count inversions on lower half

Count inversions on upper half

Count the inversions between the halves

Count the Inversions



Problem – how do we count inversions between sub problems in O(n) time?

- Solution – Count inversions while merging

1	2	3	4	7	11	12	15

5	6	8	9	10	13	14	16

Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution

Use the merge algorithm to count inversions

1	4	11	12

2	3	7	15

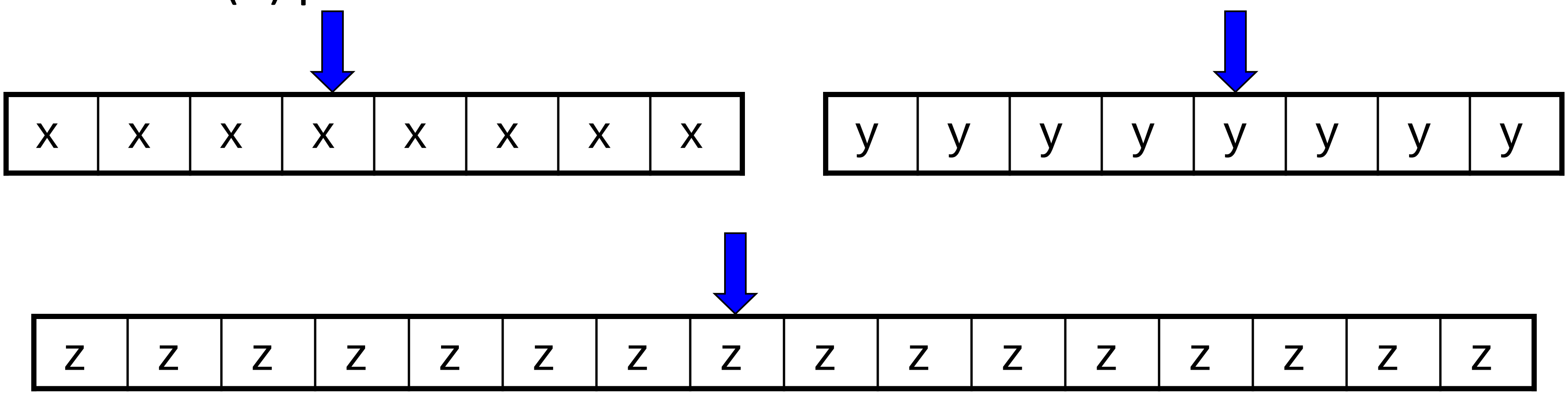
5	8	9	16

6	10	13	14

Indicate the number of inversions for each element detected when merging

Inversions

- Counting inversions between two sorted lists
  - O(1) per element to count inversions



- Algorithm summary
  - Satisfies the “Standard recurrence”
  - $T(n) = 2\,T(n/2) + cn$

Computing the Median

- Given n numbers, find the number of rank n/2
- One approach is sorting
  - Sort the elements, and choose the middle one
  - Can you do better?

Problem generalization

- *Selection*, given n numbers and an integer k, find the k-th largest

Select(A, k)

Select(A, k){

    Choose element x from A

    $S_1 = \{y \text{ in } A \mid y < x\}$

    $S_2 = \{y \text{ in } A \mid y > x\}$

    $S_3 = \{y \text{ in } A \mid y = x\}$

    if ($|S_2| >= k$)

        return Select(S_2, k)

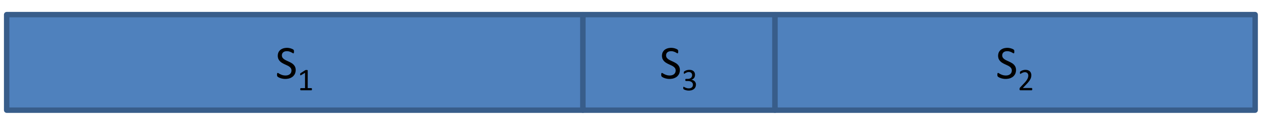
    else if ($|S_2| + |S_3| >= k$)

        return x

    else

        return Select(S_1, $k - |S_2| - |S_3|$)

}



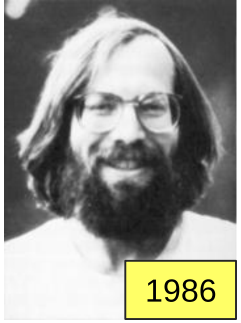
Randomized Selection

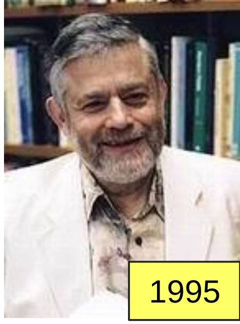
- Choose the element at random
- Analysis can show that the algorithm has expected run time O(n)

Deterministic Selection

- What is the run time of select if we can guarantee that choose finds an x such that $|S_1| < 3n/4$ and $|S_2| < 3n/4$ in $O(n)$ time

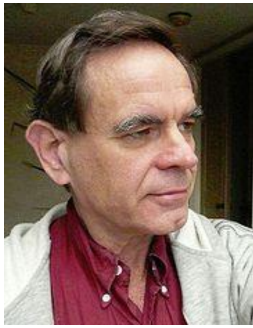
BFPRT Algorithm





- A very clever choose algorithm . . .

Split into n/5 sets of size 5
M be the set of medians of these sets
Let x be the median of M



BFPRT runtime

$|S_1| < 3n/4$, $|S_2| < 3n/4$

Split into n/5 sets of size 5
M be the set of medians of these sets
x be the median of M
Construct S_1 and S_2
Recursive call in S_1 or S_2

BFPRT Recurrence

- $T(n) <= T(3n/4) + T(n/5) + c\ n$

Prove that $T(n) <= 20\ c\ n$