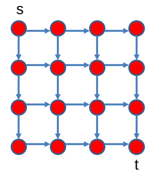
CSE 417
Algorithms and Complexity

Autumn 2020
Lecture 13
Minimum Spanning Trees

Announcements

- Homework
  - Assignment will include a sample midterm
  - Programming
    - Shortest Path and Bottleneck Paths on Grid Graphs with random edge lengths
    - What is the expected length of an s-t path?
    - What is the expected bottleneck length of an s-t path



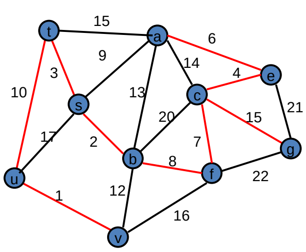
Minimum Spanning Tree

- Introduce Problem
- Demonstrate three different greedy algorithms
- Provide proofs that the algorithms work

Minimum Spanning Tree Definitions

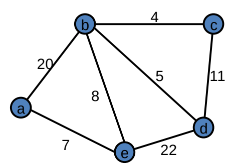
- G=(V,E) is an UNDIRECTED graph
- Weights associated with the edges
- Find a spanning tree of minimum weight
  - If not connected, complain

Minimum Spanning Tree



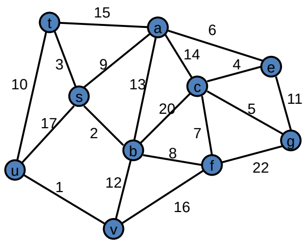
Greedy Algorithms for Minimum Spanning Tree

- Extend a tree by including the cheapest out going edge
- Add the cheapest edge that joins disjoint components
- Delete the most expensive edge that does not disconnect the graph



Greedy Algorithm 1
Prim's Algorithm

- Extend a tree by including the cheapest out going edge

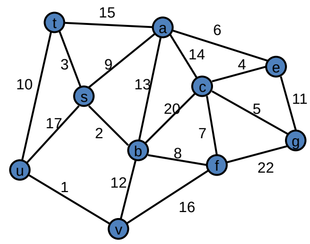


Construct the MST with Prim's algorithm starting from vertex a

Label the edges in order of insertion

Greedy Algorithm 2
Kruskal's Algorithm

- Add the cheapest edge that joins disjoint components

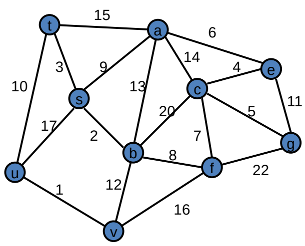


Construct the MST with Kruskal's algorithm

Label the edges in order of insertion

Greedy Algorithm 3
Reverse-Delete Algorithm

- Delete the most expensive edge that does not disconnect the graph



Construct the MST with the reverse-delete algorithm

Label the edges in order of removal

Dijkstra's Algorithm
for Minimum Spanning Trees

S = { }; d[s] = 0; d[v] = infinity for v != s

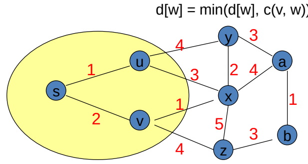
While S != V

  Choose v in V-S with minimum d[v]

  Add v to S

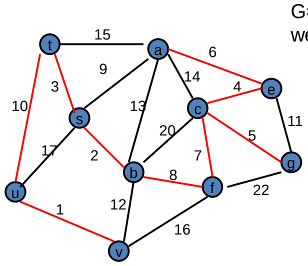
  For each w in the neighborhood of v

    d[w] = min(d[w], c(v, w))



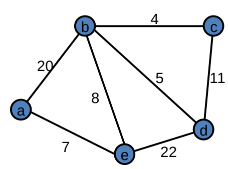
Minimum Spanning Tree

Undirected Graph G=(V,E) with edge weights



Greedy Algorithms for Minimum Spanning Tree

- [Prim] Extend a tree by including the cheapest out going edge
- [Kruskal] Add the cheapest edge that joins disjoint components
- [ReverseDelete] Delete the most expensive edge that does not disconnect the graph

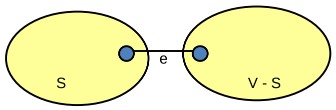


Why do the greedy algorithms work?

- For simplicity, assume all edge costs are distinct

Edge inclusion lemma

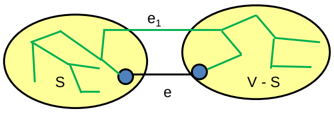
- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree of G
  - Or equivalently, if e is not in T, then T is not a minimum spanning tree



e is the minimum cost edge between S and V-S

Proof

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S



- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

Optimality Proofs

- Prim's Algorithm computes a MST
- Kruskal's Algorithm computes a MST

- Show that when an edge is added to the MST by Prim or Kruskal, the edge is the minimum cost edge between S and V-S for some set S.

Prim's Algorithm

S = { }; T = { };

while S != V

  choose the minimum cost edge
  e = (u,v), with u in S, and v in V-S

  add e to T

  add v to S

Prove Prim's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

Kruskal's Algorithm

Let $C = \{\{v_1\}, \{v_2\}, . . ., \{v_n\}\}$; $T = \{ \}$

while $|C| > 1$

Let $e = (u, v)$ with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

Replace C_i and C_j by $C_i \cup C_j$

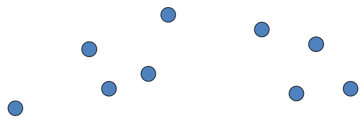
Add e to T

Prove Kruskal's algorithm computes an MST

- Show an edge e is in the MST when it is added to T

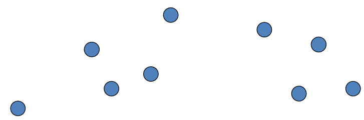
Application: Clustering

- Given a collection of points in an r-dimensional space and an integer K, divide the points into K sets that are closest together

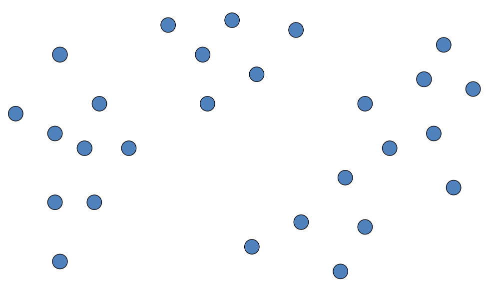


Distance clustering

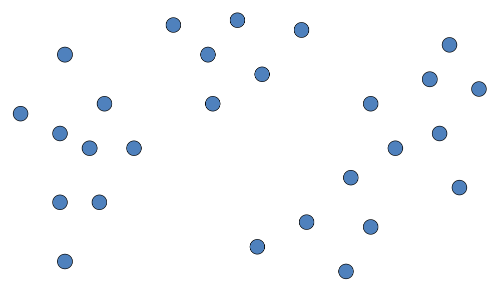
- Divide the data set into K subsets to maximize the distance between any pair of sets
  - dist $(S_1, S_2) = \min \{\text{dist}(x, y) \mid x \text{ in } S_1, y \text{ in } S_2\}$



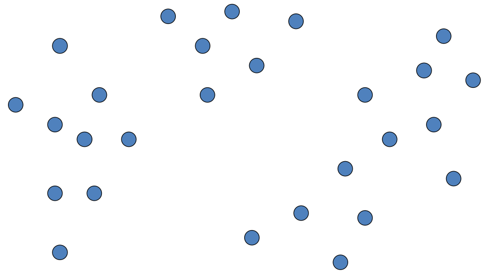
Divide into 2 clusters



Divide into 3 clusters



Divide into 4 clusters



Distance Clustering Algorithm

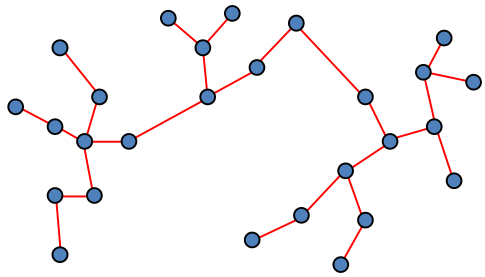
Let C = {{v_1}, {v_2},. . ., {v_n}}; T = { }

while |C| > K

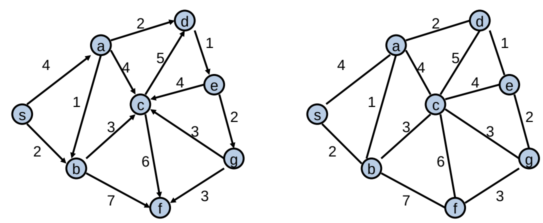
Let e = (u, v) with u in C_i and v in C_j be the minimum cost edge joining distinct sets in C

Replace C_i and C_j by C_i U C_j

K-clustering

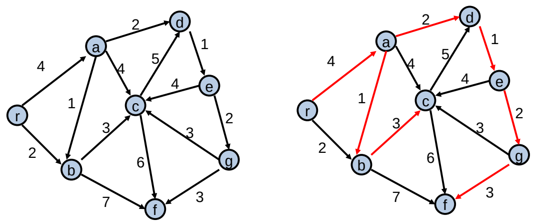


Shortest paths in directed graphs vs undirected graphs



What about the minimum spanning tree of a directed graph?

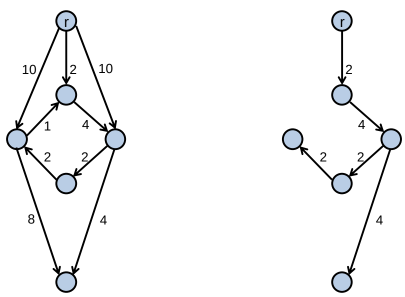
- Must specify the root r
- Branching: Out tree with root r



Assume all vertices reachable from r

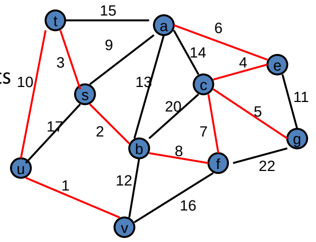
Also called an arborescence

Finding a minimum branching



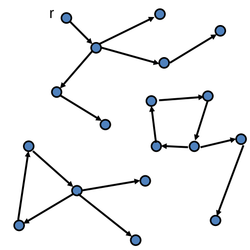
Another MST Algorithm

- Choose minimum cost edge into each vertex
- Merge into components
- Repeat until done



Idea for branching algorithm

- Select minimum cost edge going into each vertex
- If graph is a branching then done
- Otherwise collapse cycles and repeat



Finding a minimum branching

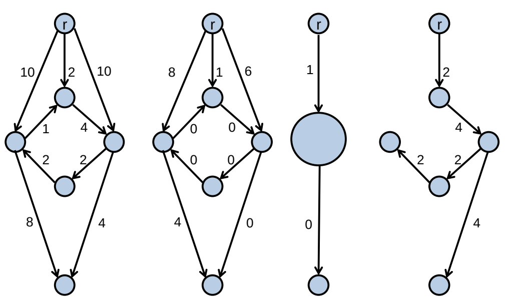
- Remove all edges going into r
- Normalize the edge weights, so the minimum weight edge coming into each vertex has weight zero

This does not change the edges of the minimum branching

Finding a minimum branching

- Consider the graph that consists of the minimum cost edge coming in to each vertex
  - If this graph is a branching, then it is the minimum cost branching
  - Otherwise, the graph contains one or more cycles
    - Collapse the cycles in the original graph to super vertices
    - Reweight the graph and repeat the process

Finding a minimum branching



Correctness Proof

Lemma 4.38 Let C be a cycle in G consisting of edges of cost 0 with r not in C. There is an optimal branching rooted at r that has exactly one edge entering C.

- The lemma justifies using the edges of the cycle in the branching
- An induction argument is used to cover the multiple levels of compressing cycles