CSE 417
Branch & Bound (pt 1)
NP-Completeness

UNIVERSITY *of* WASHINGTON

Reminders

> **HW7 is due today**

> **HW8 will be posted shortly**
>
> – network flow coding
>
> – model the problem of rounding table entries as max flow
>
> > you are given a library that solves basic max flow

Review of previous topics

- Modeling techniques
  - shortest paths (intersection of both network flows and dynamic programming)
  - binary search
  - network flows (max flow & min cost flow)
- Design techniques
  - divide and conquer
  - dynamic programming
  - **branch and bound**
    - applies to problems too hard to solve with the other techniques
    - (in particular, it applies to NP-complete problems, defined shortly...)

Outline for Today

> P and NP
> Reductions
> Some NP-complete Problems

P

- **Definition**: P is the set of problems that can be solved in polynomial time by a sufficiently large computer
  - (one with enough memory)

- Theoretical details:
  - polynomial time in the number of bits of input
    - excludes pseudo-polynomial time algorithms
  - only decision problems
    - equivalent to optimization due to binary search
  - algorithm must run on a Turing machine
    - equivalent to usual machines

P: History

- “Invented” by Jack Edmonds (1965)
  - earlier work often focused on actual running times on real machines
  - Edmonds wanted to explain the significance of his matching algorithm
    - solved general matching (harder than bipartite matching) in polynomial time
    - paper was rejected multiple times

- (Note: von Neumann and others also helped “invent” P)

P: Theory vs Practice

- Polynomial time algorithms are typically more useful in practice
  - some pseudo-poly and exponential time algorithms are useful (more later…)
  - some polynomial algorithms are not useful (e.g., $O(n^8)$)
  - in general, though, it is a good dividing line

- Need this definition to get a reasonable theory
  - want the following pieces:
    - linear time is fast
    - if function g is fast and f, which calls g as a subroutine, is fast if we count calls to g as one operation, then f is fast (*composability*)
  - those two imply polynomial time is fast

NP

- **Definition**: NP is the set of problems for which a correct answer can be verified in polynomial time
  - i.e., the problem of checking whether an answer is correct is in P
  - P $\subseteq$ NP but NP is believed to be strictly larger
  - (implies that a correct answer must be polynomial size...
    - so these are problems with small (polynomial size) proofs of correctness)

- This is not true of all problems, e.g.:
  - testing equivalence of regular expressions
  - solving (generalized) chess or go
    - proof of a winning strategy is very large

NP-Completeness

- **Definition**: Problem X is NP-hard if *any* problem in NP could be solved in polynomial time if given a function that solves X
- Shows that X is as hard as any problem in NP
- **Definition**: Problem X is NP-complete if it is in NP and is NP-hard
- NP-complete problems are the hardest problems in NP
  - can solve these problems iff P = NP
    - solving these would solve them all

NP-Completeness

- Boolean Satisfiability (SAT): given a logical formula on variables $x_1, \ldots, x_n$ using only **and**, **or**, & **not**, determine whether there is a setting of the variables to T/F so that the formula evaluates to T
  - in NP
  - short proof of correctness: give the T variables and the F variables

- **Theorem** (Cook–Levin): SAT is NP-complete
  - (see textbook for a proof)

P vs NP

- Can win $1,000,000 by proving that P ≠ NP (or P = NP)...
- Proving P = NP would be easier (if it were true)
  - just invent a polynomial time algorithm for *any* NP-complete problem
  - unfortunately, we don't think such an algorithm exists
    - since so many smart people have been trying hard for decades
- Proving P ≠ NP is deviously difficult...

P vs NP

- Can win $1,000,000 by proving that P ≠ NP (or P = NP)...
- Proving P ≠ NP is deviously difficult...
  - if P = NP, then we would have a poly time algorithm to find the proof
    - (finding a proof of certain logical statements is NP-complete)
    - unfortunately, there would be no proof in that case
  - if P ≠ NP, then we could hope for a "natural proof"
    - (formalizes the idea of how most would normally try to prove this)
    - unfortunately, Razborov & Rudich proved that such a proof would actually imply that P = NP
  - most other reasonable ideas for proofs have been ruled out
    - hard to find an approach that seems workable & hasn't been ruled out

P vs NP

- In mathematics, P vs NP is unsettled
  - though most believe they are unequal
- In physics, P ≠ NP is often taken as a physical law
  - (see recent work on black holes etc.)
  - simple version: “this Ising model must take exponential time to cool down because if not you could use it to solve NP-complete problems in poly time”
- (Actually, nearly all physicists believe BQP, not P, is the set of problems that can be solved physically...
  - these are problems solvable on quantum computers)

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Reductions

- **Definition**: Problem Y is polynomial-time reducible to X, denoted $Y \leq_P X$, if there is a polynomial time algorithm that solves Y assuming a polynomial-time subroutine for solving X.
  - algorithm makes poly(n) calls to the subroutine and does poly(n) other work
- The algorithm here is called a (Cook) "reduction"
  - show that Y is no harder than X ($Y \leq_P X$) by giving a reduction from Y to X

Reductions

> **Definition**: Problem Y is polynomial-time reducible to X, denoted $Y \leq_P X$, if there is a polynomial time algorithm that solves Y assuming a polynomial-time subroutine for solving X.

> **Re-definition**: X is NP-hard if, for *every* Y in NP, $Y \leq_P X$
>
> – i.e., X is at least as hard as any problem in NP

Reductions

- **Definition**: Problem Y is polynomial-time reducible to X, denoted $Y \leq_P X$, if there is a polynomial time algorithm that solves Y assuming a polynomial-time subroutine for solving X.

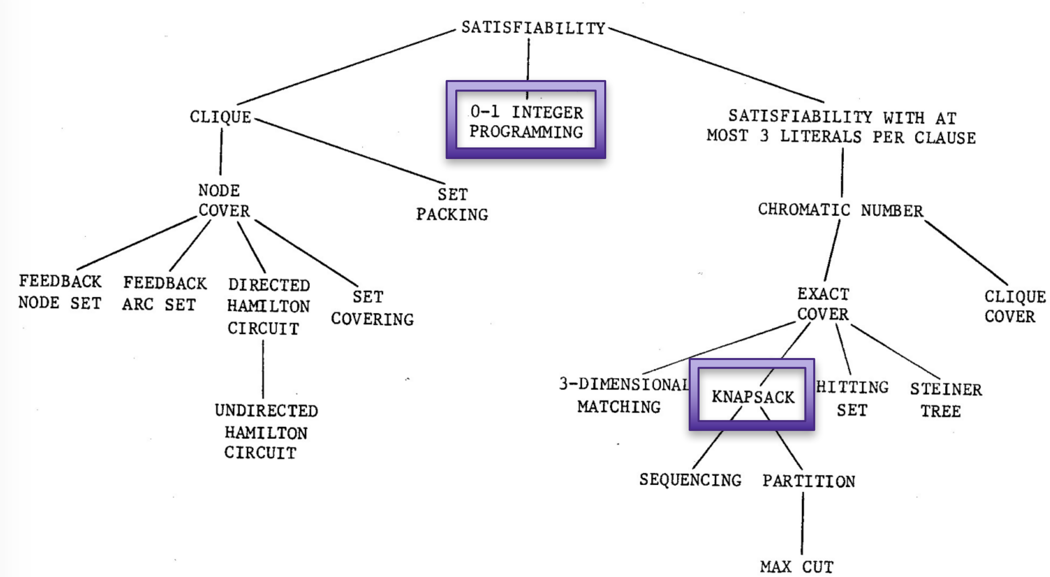
- **Warning**: do not confuse the order of X and Y!
  - a reduction from Y to X shows…
    - if you could solve X efficiently, then you could solve Y efficiently
    - so Y is no harder than X ($Y \leq_P X$)

Reductions in Theory

- Only need **one** reduction from an NP-hard problem to X to prove that X is NP-hard
  - suppose Y is NP-hard
    - i.e., $Z \leq_P Y$ for every Z in NP
  - suppose $Y \leq_P X$
  - then X is NP-hard
    - $Z \leq_P Y \leq_P X$ (so $Z \leq_P X$) for every Z in NP

Reductions in Theory

> Dick Karp popularized NP-completeness using reductions...



Dick Karp (1972)
1985 Turing Award

Reductions in Practice

- 99% of known NP-complete problems are from reductions
  - reductions seem to be much easier than direct proofs
- Reductions are a useful tool in practice
  - they let you **prove** that there is almost certainly no way to solve it efficiently
  - so you can stop trying to find an exact solution

Reductions in Practice

- Reductions are a useful tool in practice
  - they let you **prove** that there is almost certainly no way to solve it efficiently
  - so you can stop trying to find an exact solution

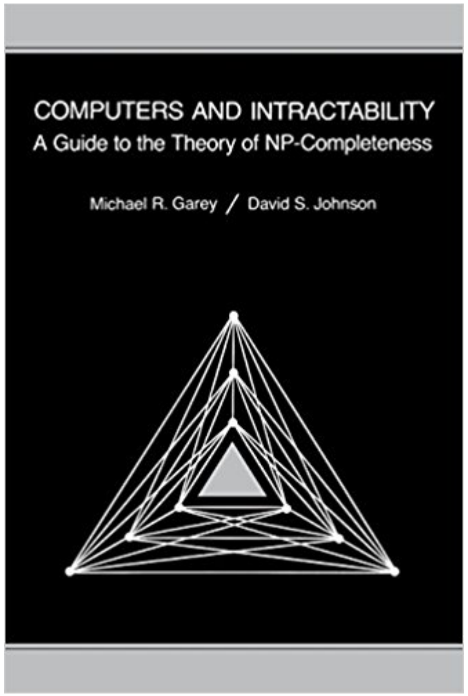
- In practice, almost every NP problem is in P or is NP-complete
  - hence, you can either find an algorithm or prove there is none
  - on the other hand..

- **Theorem** (Ladner): if P ≠ NP, then there are *infinitely many* problems that are in NP, not in P, and not NP-complete.

Garey & Johnson

- "Computers and Intractability" by Garey & Johnson contains over 300 NP-complete problems
  - can give you a quick answer for many, many problems
  - book is from 1979

- More problems have been found since then
  - see the web



More Theory About Reductions

- **Definition**: Problem Y is polynomial-time reducible to X, denoted $Y \leq_P X$, if there is a polynomial time algorithm that solves Y assuming a polynomial-time subroutine for solving X.

- **Technical Details**
  - Karp used a weaker (more difficult to achieve) notion of reduction:
    - algorithm can only make one call to the subroutine AND algorithm must simply return whatever that returns
    - i.e., algorithm constructs one problem for X to solve that has the property that it is solvable iff the Y problem given is solvable

More Theory About Reductions

- **Definition**: Problem Y is polynomial-time reducible to X, denoted $Y \leq_P X$, if there is a polynomial time algorithm that solves Y assuming a polynomial-time subroutine for solving X.

- **Technical Details**
  - Karp used a weaker (more difficult to achieve) notion of reduction
  - it is (AFAIK) still an open problem whether this difference matters
    - last I checked, every NP-completeness reduction could be made even weaker
      - (specifically, they can be done as Karp reductions in log space)
  - like the textbook, I will ignore the difference

Outline for Today

> P and NP
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NP-Complete Problems

- Have already seen some...
- Knapsack
  - pseudo-poly time algorithm shows only hard with big numbers
- 0/1 (integer) linear programming
  - reason why min cost flow problems (special case) are important
- Independent set
  - find set of k nodes in a graph with no edges between them

NP-Complete Problems

- Some are **opposites** of easy problems…
- Longest path
  - cannot simply negate edge lengths…
  - our algorithms assume no negative length cycles
- Max cut
  - can negate costs but not capacities (those were assumed ≥ 0)

NP-Compete Problems

> Both textbook and Garey & Johnson look at six problem types
  - most useful to reduce *from* ("easiest" of NP-hard problems)

Packing	independent set	clique
Covering	vertex cover	
Constraint Satisfaction	3-SAT	SAT
Sequencing	Hamiltonian cycle	TSP
Partitioning	3D matching	
Numerical	partition	subset sum, knapsack

Packing Problems

- **Independent Set**: Given graph G and number k, find a subset of k nodes such that no two are connected by an edge
- **Clique**: Given graph G and number k, find a subset of k nodes such that every pair are connected by an edge
- Reductions (Independent Set $\equiv_P$ Clique):
  - Let G' be the opposite graph:
    - N' = N
    - (u,v) in E' iff (u, v) not in E
  - nodes are independent in G iff they are clique in G'

Covering Problems

- **Vertex Cover**: Given graph G and number k, find a subset of k nodes such that every edge is adjacent to at least one of them

- Reduction (Independent Set $\equiv_P$ Vertex Cover):
  - subset S is independent iff V – S is a vertex cover:
    - S is independent **iff** for every (u,v) in E, either u not in S or v not in S **iff** for every (u,v) in E, either u in V – S or v in V – S **iff** V – S is covering
  - reduction to vertex cover:
    - call vertex cover with k replaced by n – k
  - reduction to independent set: same

Constraint Satisfaction Problems

- **SAT**: Given a logical formula on variables x_1, ..., x_n using only **and**, **or**, & **not**, determine whether there is a setting of the variables to T/F so that the formula evaluates to T

- **3-SAT**: As above, but formula is of the form "t_1 **and** t_2 ... **and** t_m", where each t_i is of the form "f_{i1} **or** f_{i2} **or** f_{i3}", where each f_{ij} is either "x_k" or "**not** x_k" for some k
  - e.g.: ((not x_1) or x_2 or x_3) and
    (x_1 or (not x_2) or x_3) and
    ((not x_1) or (not x_2) or (not x_3))

Constraint Satisfaction Problems

- **SAT**: Given a logical formula on variables $x_1, \ldots, x_n$ using only **and**, **or**, & **not**, determine whether there is a setting of the variables to T/F so that the formula evaluates to T
  - Cook proved directly that this is NP-complete

- **3-SAT**: As above, but formula is of the form "t_1 **and** t_2 … **and** t_m", where each t_i is of the form "f_{i1} **or** f_{i2} **or** f_{i3}", where each f_{ij} is either "x_k" or "**not** x_k" for some k
  - can see that 3-SAT $\leq_P$ SAT because former is special case
  - requires work to show the opposite (will skip details)

Sequencing Problems

- **Hamiltonian Cycle**: Given a graph G, find a cycle that visits every node exactly once (a “simple” cycle of length n)
- **Traveling Salesperson Problem (TSP)**: Given weighted graph G and number v, find a Hamiltonian cycle of length at most v
- Reduction (Hamiltonian Cycle $\leq_P$ TSP):
  - take the weight of each edge to be 1
  - find a cycle of length n

Partitioning Problems

> **3D Matching**: Given disjoint sets X, Y, Z and a set of M of triples of the form (x,y,z), with x in X, y in Y, and z in Z and a number k, find a set of k triples with no x's, y's, or z's in common

– could call this "tri-partite matching"

> Reduction (bipartite matching $\leq_P$ 3D matching):

– node set is split into X and Y

– let Z be the set of edges

– triple for each edge (u, v, (u,v))

  > no triples have the same Z part, so intersection means u or v is same

– note: this does not show that bipartite matching is NP-hard!

Numerical Problems

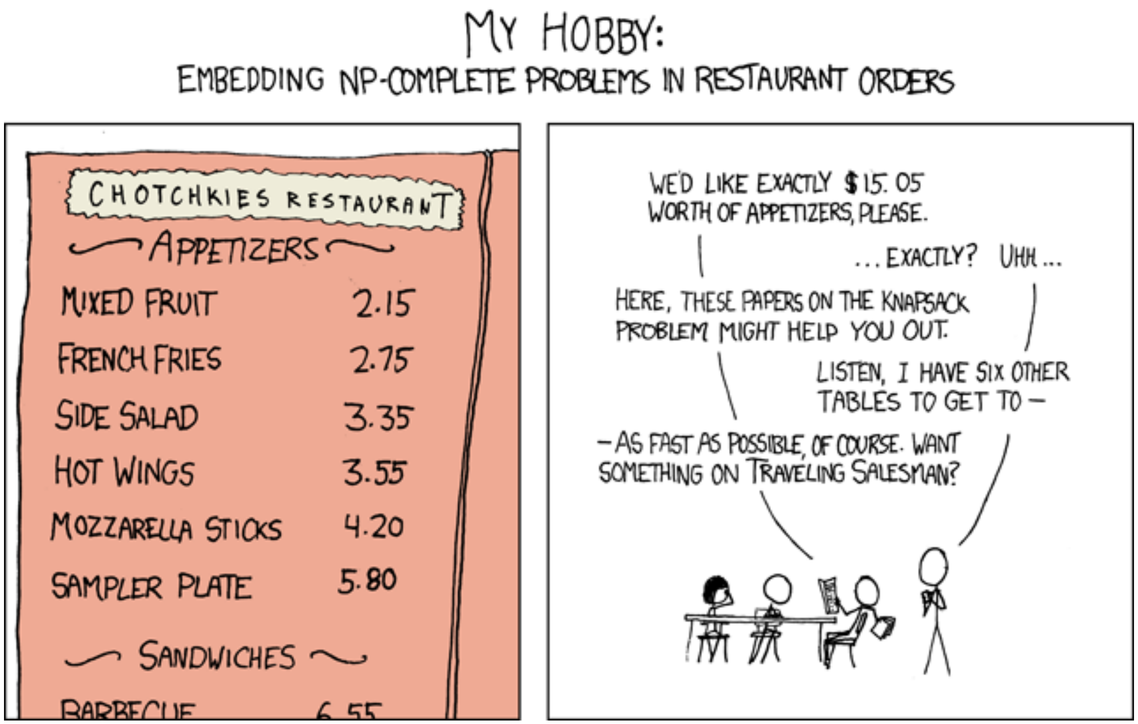
> **Knapsack**: Given items of the form (w_i, v_i) and a number W, find the largest total value of any subset of total weight at most W

> **Subset Sum**: Given weights $w_1, \ldots, w_n$ and a number W, find a subset whose total weight is exactly W

> Reduction (Subset Sum $\leq_P$ Knapsack):
  - choose the values equal to the weights
  - Knapsack gives the largest sum of weights ≤ W
  - just check if it equals W

https://xkcd.com/287/

Numerical Problems



Numerical Problems

- **Partition**: Given set $W = \{w_1, \ldots, w_n\}$ of weights, find a subset S such that total weight of S is total weight of W – S
  - i.e., does it split into two parts of equal weights?

- Reduction (Subset Sum $\leq_P$ Partition):
  - add two extra weights: (sum of weights) + W and 2 x (sum of weights) – W
  - total weight of all items is now 4 x (sum of weights)
  - two new elements cannot be in the same side of partition