CSE 417
Dynamic Programming (pt 6)
Optimizations

UNIVERSITY *of* WASHINGTON

Reminders

- **HW5 due today**

- **HW6 will be posted shortly**
  - solve a Knapsack-type problem
  - many additional wrinkles
  - will need some optimization

Dynamic Programming Review

optimal substructure: (small) set of solutions, constructed from solutions to sub-problems that is guaranteed to include the optimal one

> Apply the steps...

1. Describe solution in terms of solution to *any* sub-problems
2. Determine all the sub-problems you'll need to apply this recursively
3. Solve every sub-problem (once only) in an appropriate order

> Key question:

1. Can you solve the problem by combining solutions from sub-problems?

> Count sub-problems to determine running time

– total is number of sub-problems times time per sub-problem

Review From Previous Lectures

- **Previously:**
  - Find opt substructure by considering how the opt solution could use the last input.
    - Given multiple inputs, consider how opt uses last of either or both
    - Given clever choice of sub-problems, find opt substructure by considering new options
  - Alternatively, consider the shape of the optimal solution in general, e.g., tree structured
- **Longest Common Subsequence / Edit Distance:**
  - opt either uses last of first, last of second, both, or neither
  - edit distance is a generalization that allows substitutions & has arbitrary costs
- **Pattern Matching**
  - like max sub-array sum, switch to finding longest ending at each i
  - consider how last char of pattern matches: several cases

Outline for Today

- Space Considerations
- Divide & Conquer
- Sparseness
- Monotonicity

Simple Space Optimization

- Many DP algorithms only need to keep a small part of the table…
  - Knapsack only uses 1 .. n-1 with limit V ≤ W for 1 .. n with limit W
    - keep just the column of solutions for 1 .. n-1
    - O(W) space
  - All-pairs shortest path uses solution with intermediate nodes 1 .. k-1 for 1 .. k
    - keep all pairs distance
    - $O(n^2)$ space (rather than $O(n^3)$)
    - likewise for single-shortest shortest path with negative weights

Simple Space Optimization

- Many DP algorithms only need to keep a small part of the table...
  - Knapsack only uses 1 .. n-1 with limit V ≤ W for 1 .. n with limit W
  - All-pairs shortest path uses solution with intermediate nodes 1 .. k-1 for 1 .. k
  - Longest common subsequence only needs prefixes of $a_1, .., a_n$ and $b_1, .., b_m$ that are at most 1 shorter
    - can just keep the previous row / column (whichever is smaller)
    - O(1) space for pattern matching
  - Not just for 2D or 3D tables:
    - max sub-array sum only needs 1 .. n-1 for 1 .. n also
    - document layout in TeX only needs 1 .. j such that the words j+1 .. n-1 can fit on *one line*
    - O(1) space

Simple Space Optimization

> Unfortunately, this does apply to many of the others…
  – weighted interval scheduling needs (potentially) every prefix of 1 .. n
  – optimal breakout trades potentially needs every prefix of 1 .. n
  – all problems on trees:
    > optimal BSTs, matrix chain multiplication, optimal polygon triangulation
    > need to consider every choice of root…

	A	B	C	D	E	F
1		a	b	c	d	e
2	a	1	4	10	18	F2
3	b		2	7	15	26
4	c			3	10	20
5	d				4	13
6	e					5

Simple Space Optimization

- Unfortunately, this does apply to many of the others...
  - weighted interval scheduling needs (potentially) every prefix of 1 .. n
  - optimal breakout trades potentially needs every prefix of 1 .. n
  - all problems on trees:
    - optimal BSTs, matrix chain multiplication, optimal polygon triangulation
    - need to consider every choice of root

- No obvious way to improve space use for these

Finding Solutions

- Simple space optimizations only work if we just want **opt value**
  - saw that our formulas for opt value didn't need earlier parts of the table
  - sometimes that is fine: max sub-array sum, edit distance, pattern matching

- Seemingly need whole table to find the **solution** achieving opt
  - often need solution: opt BST, knapsack, shortest path, etc.

Finding Solution Example

> Find longest common subsequence of
  A = [1, 2, 1, 5, 4, 3]
  B = [2, 1, 3, 2, 1, 4]

> Recall our formula

opt value for 1, …, i and 1, …, j =
  max(opt value for 1, …, i-1 and 1, …, j,
    opt value for 1, …, i and 1, …, j-1,
    (opt value for 1, …, i-1 and 1, …, j-1) + (1 if $a_i = b_j$ else 0))

up one row
left one column
up one & left one

Finding Solution Example

no matches with empty lists...

		2	1	3	2	1	4
	0	0	0	0	0	0	0
1	0						
2	0						
1	0						
5	0						
4	0						
3	0						

i

j

Finding Solution Example

		2	1	3	2	1	4
	0	0	0	0	0	0	0
1	0	0					
2	0	1					
1	0	1					
5	0	1					
4	0	1					
3	0	1					

i

j

match [2]...

Finding Solution Example

			2	1	3	2	1	4
		0	0	0	0	0	0	0
	1	0	0	1				
	2	0	1	1				
i	1	0	1	2				
	5	0	1	2				
	4	0	1	2				
	3	0	1	2				

j

match [2, 1]...

Finding Solution Example

		2	1	3	2	1	4
	0	0	0	0	0	0	0
1	0	0	1	1			
2	0	1	1	1			
1	0	1	2	2			
5	0	1	2	2			
4	0	1	2	2			
3	0	1	2	3			

i

j

match [2, 1, 3]...

Finding Solution Example

			2	**1**	**3**	**2**	**1**	**4**
		0	0	0	0	0	0	0
	1	0	0	1	1	1	2	2
	2	0	1	1	1	2	2	2
i	**1**	0	1	2	2	2	3	3
	5	0	1	2	2	2	3	3
	4	0	1	2	2	2	3	4
	3	0	1	2	3	3	3	4

j

Finding Solution Example

- Table tells us the value of the optimal solution
- Walk **backward** through table to find solution achieving that value
- Each option from formula describes how the opt solution uses the last element(s):

solution doesn't use a_i

solution doesn't use b_j

a_i matched to b_j

opt value for 1, …, i and 1, …, j =
    max(opt value for 1, …, i-1 and 1, …, j,
        opt value for 1, …, i and 1, …, j-1,
        (opt value for 1, …, i-1 and 1, …, j-1) + (1 if $a_i = b_j$ else 0))

Finding Solution Example

		2	**1**	**3**	**2**	**1**	**4**
	0	0	0	0	0	0	0
1	0	0	1	1	1	2	2
2	0	1	1	1	2	2	2
1	0	1	2	2	2	3	3
5	0	1	2	2	2	3	3
4	0	1	2	2	2	3	4
3	0	1	2	3	3	3	4

i

j

Finding Solution Example

		2	1	3	2	1	4
	0	0	0	0	0	0	0
1	0	0	1	1	1	2	2
2	0	1	1	1	2	2	2
1	0	1	2	2	2	3	3
5	0	1	2	2	2	3	3
4	0	1	2	2	2	3	4
3	0	1	2	3	3	3	4

i

j

More Space Optimizations

- Simple space optimizations only work if we just want **opt value**
  - saw that our formulas for opt value didn't need earlier parts of the table
- Seemingly need whole table to find the **solution** achieving opt
  - not an issue for max-subarray sum or pattern matching
  - BUT is for *all of the others*
- Particularly important for computational biology
  - DNA similarity (edit distance) and RNA secondary structure
  - inputs can be huge

Outline for Today

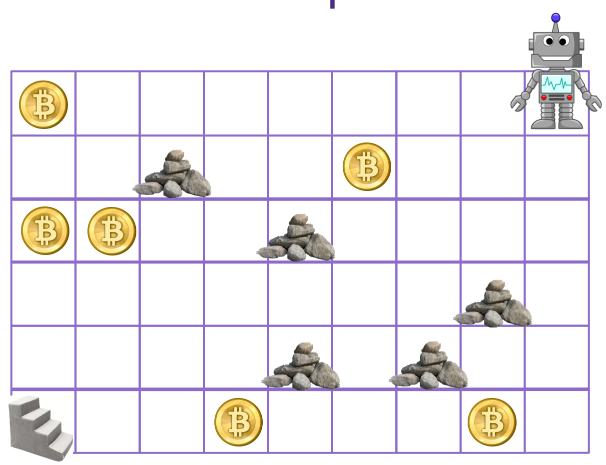
- > Space Considerations
- > Divide & Conquer
- > Sparseness
- > Monotonicity

Finding Solutions by D&C

- Simple optimization allows us to save a lot of space
  - e.g., go from O(nm) to O(n) or O(m)
- BUT we seem to lose the ability to get the actual solution
  - we usually need that
- In fact, we can get the best of both worlds using prior technique: **divide & conquer**

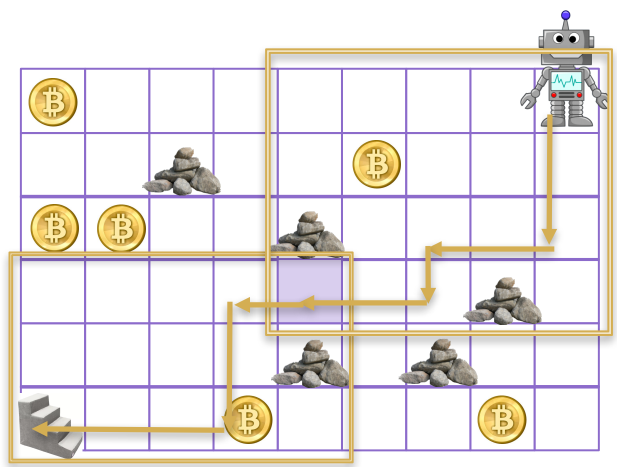
Finding Solutions by D&C

- Solution comes from the path along which we achieve opt value
- It’s not obvious what sub-problems would be useful…



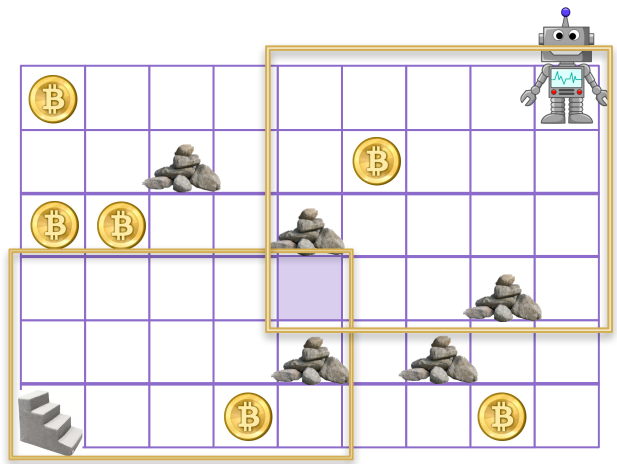
Finding Solutions by D&C

- It's not obvious what sub-problems would be useful...
  - these two would work **IF** the optimal solution goes through purple square
  - but there is no way to know if it does



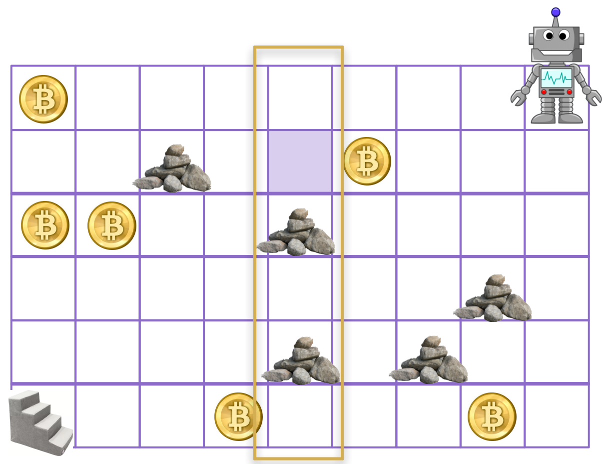
Finding Solutions by D&C

- It's not obvious what sub-problems would be useful…
  - these two would work **IF** the optimal solution goes through purple square…
  - but there is no way to know if it does (in fact, it does not in this case)



Finding Solutions by D&C

> **Problem**: Find out where the opt solution goes through middle col
  - if it crosses at (i, m/2), then paths from sub-problem on 1 .. i and 1 .. m/2 and sub-problem i .. n and m/2 .. m <u>concatenate</u> to give us the full path



Finding Solutions by D&C

- **Problem**: Find out where the opt solution goes through middle col
  - if it crosses at (i, m/2), then paths from sub-problem on 1 .. i and 1 .. m/2 and sub-problem i .. n and m/2 .. m <u>concatenate</u> to give us the full path

- D&C algorithm:
  - find index i where the opt path passes through (i, m/2)
  - recursively find opt path on 1 .. i and 1 .. m/2
  - recursively find opt path on i .. n and m/2 .. m
  - return concatenation of those two paths which meet at (i, m/2)

Finding Opt Path at Midpoint

- Find opt values with DP
- Find opt path with D&C assuming we can solve:

Problem: Find out where the opt solution goes through middle col

- **Q**: How do we solve this problem?
- **A**: dynamic programming

Finding Opt Path at Midpoint

> **Idea**: compute not just the opt value but also where the opt solution passed through the midpoint

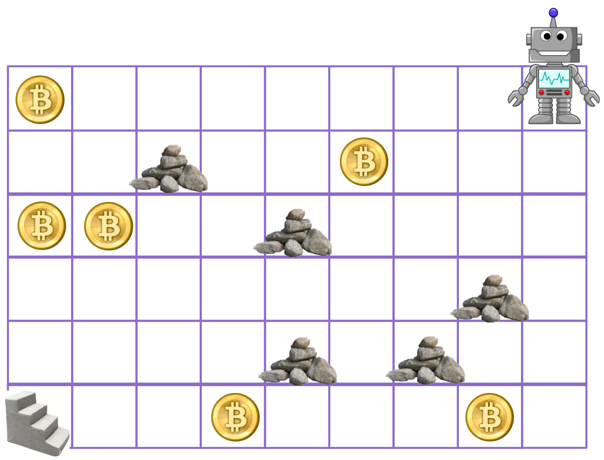
opt-mid at (i,j) =

undefined	if $j < m/2$	(don't know yet)
i	if $j = m/2$	
opt-mid(i', j')	if $j > m/2$	

where opt solution at (i, j) goes through (i', j')

- we know opt solution goes through (i', j') — some sub-problem
- we know opt solution from (i', j') crosses mid at opt-mid(i', j')

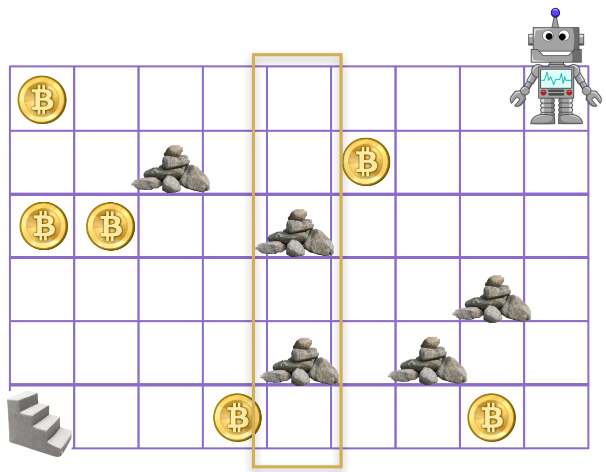
Finding Opt Path at Midpoint



opt values

2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

Finding Opt Path at Midpoint

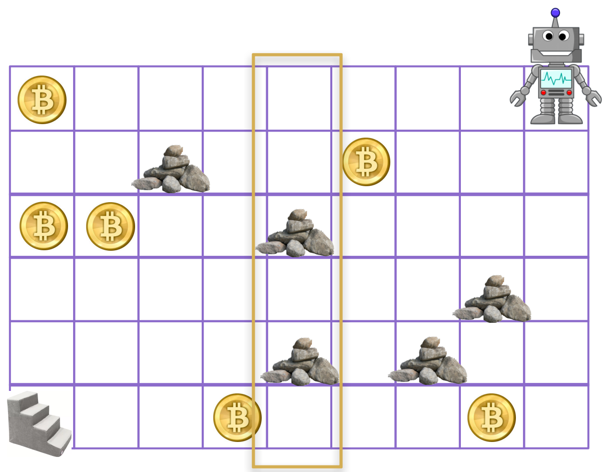


opt values

2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

Finding Opt Path at Midpoint



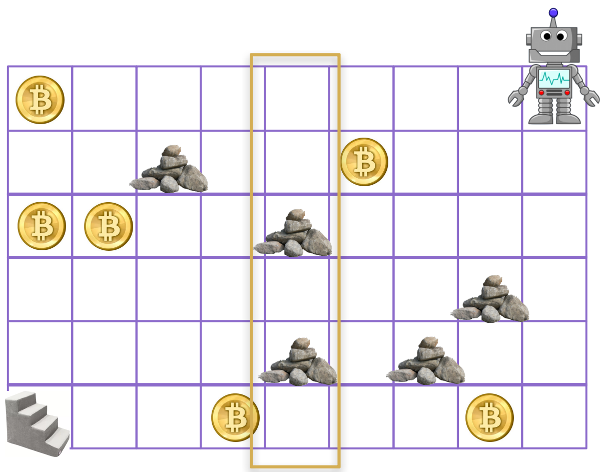
opt values

2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6				
				5				
				4				
				3				
				2				
				1				

Finding Opt Path at Midpoint



opt values

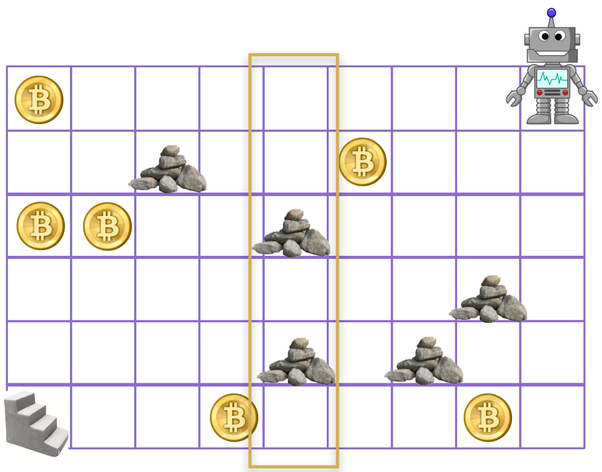
2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5			
				5	5			
				4	3			
				3	3			
				2	1			
				1	1			

opt path from purple spot goes left

Finding Opt Path at Midpoint



opt values

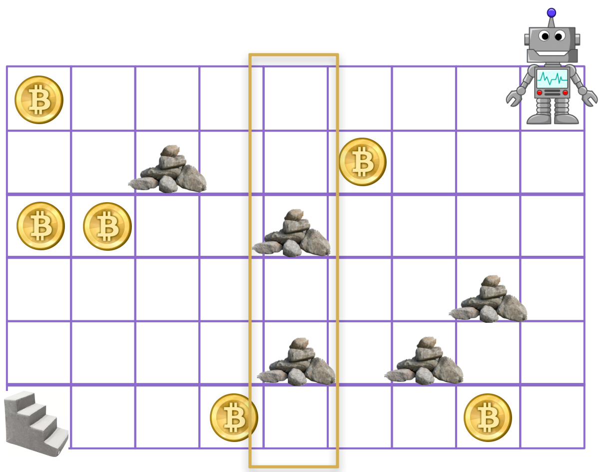
2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5			
				5	5			
				4	3			
				3	3			
				2	1			
				1	1			

opt path from purple spot goes down

Finding Opt Path at Midpoint



opt values

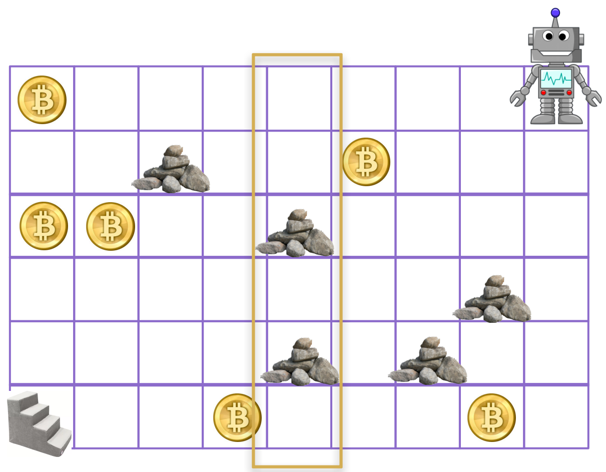
2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5			
				5	5			
				4	3			
				3	3			
				2	1			
				1	1			

opt path from purple spot goes left or down

Finding Opt Path at Midpoint



opt values

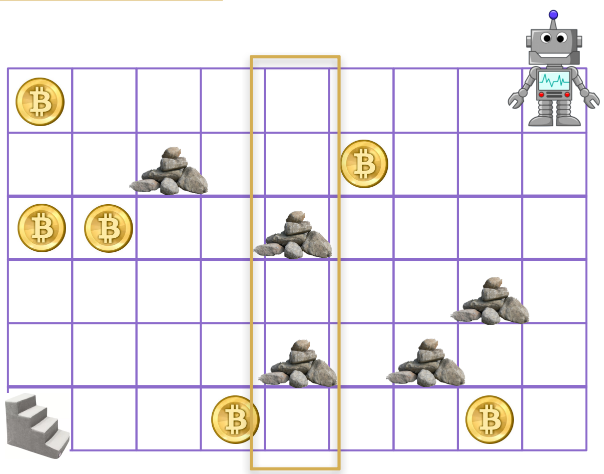
2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5			
				5	5			
				4	3			
				3	3			
				2	1			
				1	1			

opt path from purple spot goes down

Finding Opt Path at Midpoint



opt values

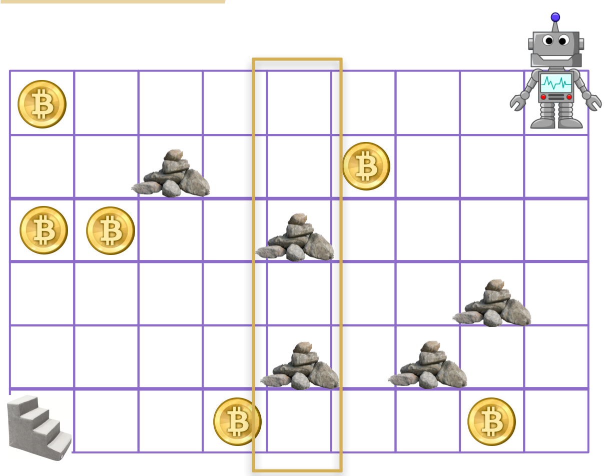
2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5			
				5	5			
				4	3			
				3	3			
				2	1			
				1	1			

opt path from purple spot goes down

Finding Opt Path at Midpoint



opt values

2	2	2	2	2	3	3	3	3
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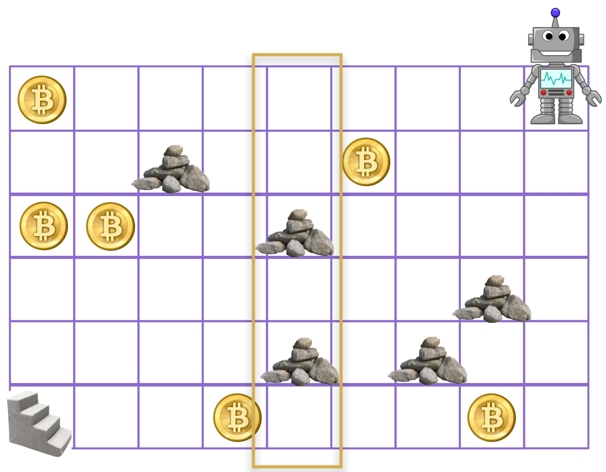
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5			
				5	5			
				4	3			
				3	3			
				2	1			
				1	1			

opt path from purple spot goes down

Finding Opt Path at Midpoint



opt values

2	2	2	2	2	3	3	3	3
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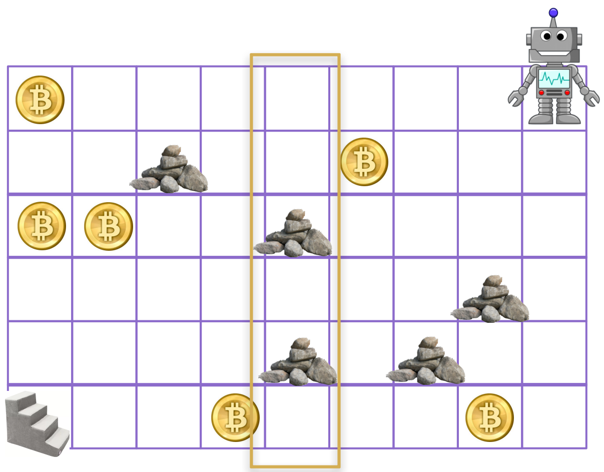
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5	5		
				5	5	5		
				4	3	3		
				3	3	3		
				2	1			
				1	1	1		

Finding Opt Path at Midpoint



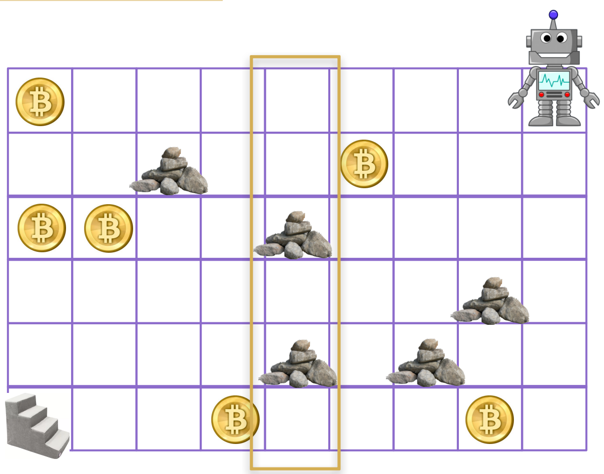
opt values

2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5	5	5	
				5	5	5	5	
				4	3	3	3	
				3	3	3		
				2	1		1	
				1	1	1	1	

Finding Opt Path at Midpoint



opt values

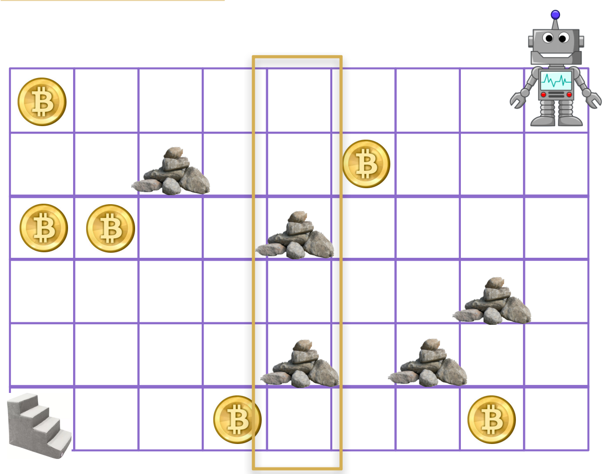
2	2	2	2	2	3	3	3	3
1	2	0	2	2	3	3	3	3
1	2	2	2	0	1	1	1	2
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5	5	5	
				5	5	5	5	
				4	3	3	3	1
				3	3	3		1
				2	1		1	1
				1	1	1	1	1

opt path from purple spot goes down not left

Finding Opt Path at Midpoint



opt values

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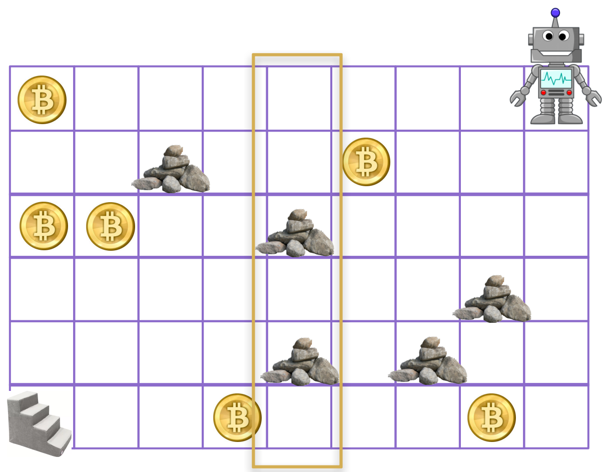
0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5	5	5	
				5	5	5	5	5
				4	3	3	3	1
				3	3	3		1
				2	1		1	1
				1	1	1	1	1

opt path from purple spot goes left not down

Finding Opt Path at Midpoint



opt values

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0	0	0	1	1	1	1	0	2
0	0	0	1	0	1	0	2	2
0	0	0	1	1	1	1	2	2

opt mid-point

				6	5	5	5	5
				5	5	5	5	5
				4	3	3	3	1
				3	3	3		1
				2	1		1	1
				1	1	1	1	1

Finding Solutions by D&C: Summary

> Compute opt-mid in (at most) same amount of time as opt value
  - record not only which value in the set is the min/max but also which one it was
  - only changes total running time by a constant factor

> If we can compute the opt value in time t(n) and space s(n), we can compute the opt solution in time O(t(n) log n) and space s(n)

can be O(t(n))... see master thm

> Can still save space by keeping only last row / col, at the cost of O(log n) factor in running time

Outline for Today

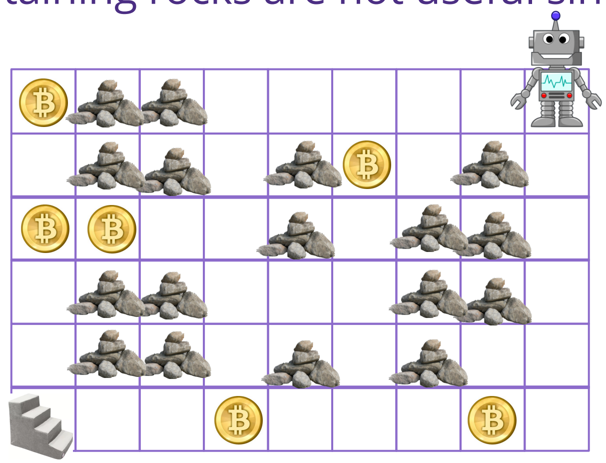
> Space Considerations
> Divide & Conquer
> Sparseness
> Monotonicity

Sparse Tables

- Sometimes, only a fraction of the matrix has useful values
- We have seen one example of this already...

Sparse Tables

- Sometimes, only a fraction of the matrix has useful values
- We have seen one example of this already…
  - entries containing rocks are not useful since they cannot be used on a path



Sparse Tables

- Sometimes, only a fraction of the matrix has useful values
  - we have seen one example of this already… the robot problem
  - entries containing rocks are not useful since they cannot be used on a path
- Save space (and time) by not storing (or computing) these values

Sparse Tables

- Sometimes, only a fraction of the matrix has useful values
  - we have seen one example of this already… the robot problem
  - entries containing rocks are not useful since they cannot be used on a path
- The robot problem is trying to find the best path in a **directed acyclic graph** (DAG)
  - solve that by DP even with negative weights (since no cycles)
  - algorithm runs in O(n + m) time for n node, m edge graph
    - (note that we need to process nodes in the right order)
- Storing data for nodes save space/time in *sparse* graphs

Sparseness in LCS

- Table entries can also be useless in other ways...

Sparseness in LCS

only place where you add elements to solution...

> In longest common subsequence, the only “interesting” table entries are those for (i, j) with $a_i = b_j$

		2	1	3	2	1	4
	0	0	0	0	0	0	0
1	0	0	1	1	1	2	2
2	0	1	1	1	2	2	2
1	0	1	2	2	2	3	3
5	0	1	2	2	2	3	3
4	0	1	2	2	2	3	4
3	0	1	2	3	3	3	4

solution is a path that only moves in up/left direction with each step

Sparseness in LCS

- In longest common subsequence, the only "interesting" table entries are those for (i, j) with $a_i = b_j$

- Could solve the problem by constructing a graph whose nodes are the interesting entries
  - if there are K such entries, we have K nodes and $O(K^2)$ edges
  - DAG algorithm finds the best path in $O(K^2)$ time
  - total time to build and solve is $O(n + m + K^2)$
    - put each sequence into a hash table… (or use sort & binary search)
  - this is faster provided that $K \ll (nm)^{0.5}$

Sparseness in LCS

- In longest common subsequence, the only “interesting” table entries are those for (i, j) with $a_i = b_j$
- Non-interesting entries are just the max of the values left & above
  - applied recursively…
  - each such entry is the maximum of the interesting entries above / left
  - this is a slow way to solve the problem if there are few interesting entries

Sparseness in Knapsack

> Knapsack uses a lot of memory… would be nice to have sparseness

> As you'll see in HW6, however, that is not what happens
  - column for prefix 1 .. j stores optimal solution for any **subset** of items 1– j
  - there are 2^j such subsets, most of which have distinct values
  - the columns quickly become dense

> One special case: all numbers have a common divisor
  - then every sum is divisible by that number as well
  - so there is no need to store table entries for other numbers

Sparseness Summary

- DP solves shortest path on DAGs in O(n+m) time
  - (more details on how to order the nodes in the general case…)
- Can use that to speed up the computation when the entries used in the solution are limited to a small subset of "interesting" ones
  - for LCS, the optimal path goes through non-interesting entries BUT those do not contribute anything to the actual solution
  - non-interesting entries indicate an element was *not used*
- Don't expect sparseness when solutions are subsets
  - number of possible subsets grows very quickly

Outline for Today

- Space Considerations
- Divide & Conquer
- Sparseness
- Monotonicity

Time Optimizations

- There are also some general techniques for improving the time complexity of DP algorithms in certain cases
  - see, e.g., Yao on “quadrangle inequalities”
  - see, e.g., Galil & Park on exploiting convexity / concavity
- We will look at the “Knuth optimization”
  - probably the most famous of these

Knuth Optimization

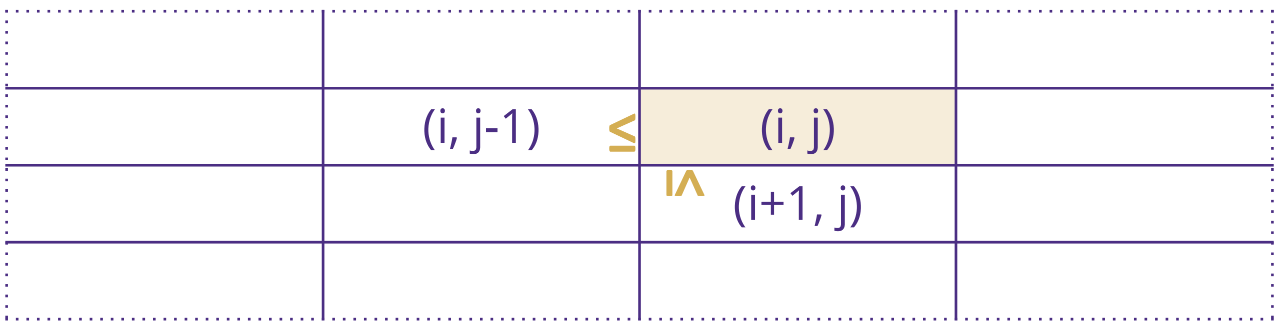
> This technique applies where our matrix of solutions OPT satisfies:

$$\text{OPT}[i, j-1] \leq \text{OPT}[i, j] \leq \text{OPT}[i+1, j] \quad \text{for all } i, j$$

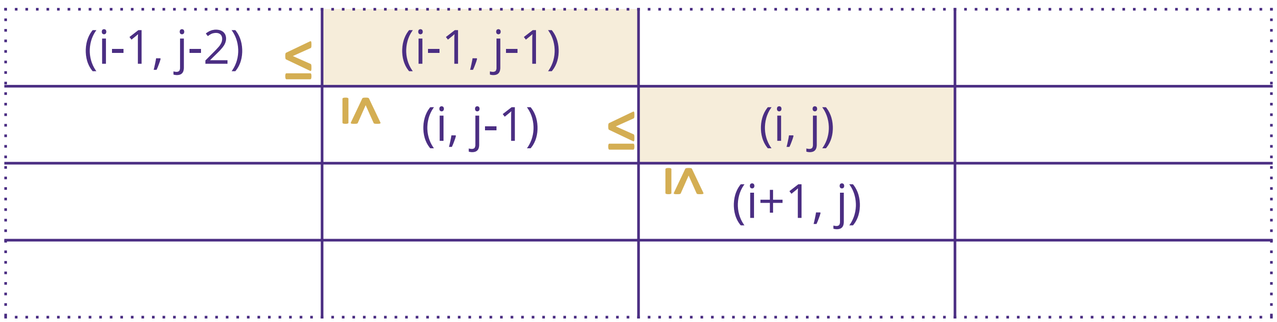
> This holds for, e.g., the optimal binary search tree problem
  - (see Knuth's "Art of Computer Programming" volume 3)
  - this is in no way obvious!
    > (in fact, it's not even true as stated...
    true claim is about which element is the root, not the optimal value)

> With that, the optimization is straightforward...

Knuth Optimization



Knuth Optimization



Knuth Optimization

(i-1, j-2) ≤	(i-1, j-1)		
	(i, j-1) ≤	(i, j)	
		(i+1, j) ≤	(i+1, j+1)
			(i+2, j+1)

... ≤ OPT[i-1, j-1] ≤ OPT[i, j] ≤ OPT[i+1, j+1] ≤ ...

and each is limited to a **non-overlapping** range

Knuth Optimization

- Recall, in optimal BST problem, sub-problems are ranges i .. j
- Base cases are ranges of size 1 (i .. i)
- Solve problem on i .. j using only subranges of i .. j
  - in particular, we need only **shorter** ranges to solve **longer** ones
  - ranges of the same length are along the same diagonal…

Knuth Optimization

size 1...

1 .. 1						
	2 .. 2					
		3 .. 3				
			4 .. 4			
				5 .. 5		
					6 .. 6	
						7 .. 7

Knuth Optimization

size 2...

1 .. 1	1 .. 2					
	2 .. 2	2 .. 3				
		3 .. 3	3 .. 4			
			4 .. 4	4 .. 5		
				5 .. 5	5 .. 6	
					6 .. 6	6 .. 7
						7 .. 7

Knuth Optimization

size 3…

1 .. 1	1 .. 2	1 .. 3				
	2 .. 2	2 .. 3	2 .. 4			
		3 .. 3	3 .. 4	3 .. 5		
			4 .. 4	4 .. 5	4 .. 6	
				5 .. 5	5 .. 6	5 .. 7
					6 .. 6	6 .. 7
						7 .. 7

Knuth Optimization

1 .. 1	1 .. 2	1 .. 3	1 .. 4	1 .. 5	1 .. 6	1 .. 7
	2 .. 2	2 .. 3	2 .. 4	2 .. 5	2 .. 6	2 .. 7
		3 .. 3	3 .. 4	3 .. 5	3 .. 6	3 .. 7
			4 .. 4	4 .. 5	4 .. 6	4 .. 7
				5 .. 5	5 .. 6	5 .. 7
					6 .. 6	6 .. 7
						7 .. 7

Knuth Optimization

> Recall, in optimal BST problem, sub-problems are ranges i .. j
  - base cases are ranges of size 1 along the main diagonal
  - solve remaining problems along increasing upper diagonals

> Along each **diagonal**, values can only increase moving down / right

> AND each lies in a *non-overlapping* range
  - OPT[i-1, j-2] to OPT[i, j-1]
  - OPT[i, j-1] to OPT[i+1, j]
  - OPT[i+1, j] to OPT[i+2, j+1]
  - …

Knuth Optimization

(i-1, j-2) ≤	(i-1, j-1)		
	≤ (i, j-1) ≤	(i, j)	
		≤ (i+1, j) ≤	(i+1, j+1)
			≤ (i+2, j+1)

... ≤ OPT[i-1, j-1] ≤ OPT[i, j] ≤ OPT[i+1, j+1] ≤ ...

and each is limited to a **non-overlapping** range

Knuth Optimization

- Recall, in optimal BST problem, sub-problems are ranges i .. j
  - base cases are ranges of size 1 along the main diagonal
  - solve remaining problems along increasing upper diagonals

- Along each diagonal, values can only increase
  AND each lies in a *non-overlapping* range

- If all the values are in a range of size O(n),
  we can compute them all in O(n) time
  - reduces the overall running time from $O(n^3)$ to $O(n^2)$