CSE 417
Dynamic Programming (pt 5)
Multiple Inputs

UNIVERSITY *of* WASHINGTON

Reminders

> HW5 due Wednesday

Dynamic Programming Review

optimal substructure: (small) set of solutions, constructed from solutions to sub-problems that is guaranteed to include the optimal one

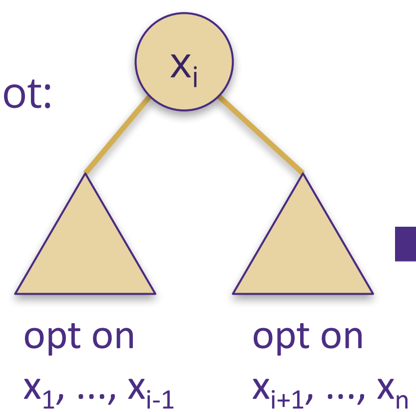
> Apply the steps...
1. Describe solution in terms of solution to *any* sub-problems
2. Determine all the sub-problems you'll need to apply this recursively
3. Solve every sub-problem (once only) in an appropriate order

> Key question:
1. Can you solve the problem by combining solutions from sub-problems?

> Count sub-problems to determine running time
- total is number of sub-problems times time per sub-problem

Review From Previous Lectures

- **Previously:**
  - Find opt substructure by considering how the opt solution could use the last input.
  - Given clever choice of sub-problems, find opt substructure by considering new options
- **Tree Structure:**
  - Sub-problems are left and subtrees
  - opt value = min cost of tree over choices of root:
  - Problems:
    - optimal binary search trees
    - matrix chain multiplication
    - optimal polygon triangulation (HW5)



Outline for Today

- > Multiple Inputs Generally
- > Longest Common Subsequence
- > Edit Distance
- > Pattern Matching

Multiple Inputs

- Have mainly looked at problems whose input is a list of items
- Now, we will look at problems with multiple lists of inputs...
- Can still use the same heuristic to find optimal sub-substructure: consider how the optimal solution might use the last element

Multiple Inputs

- Can still use the same heuristic to find optimal sub-substructure: consider how the optimal solution might use the last element

- Difference is that there are multiple last elements
  - the last one from each list

- To use the heuristic, consider how opt uses any of the last elements...
  - could think about just one or all of them simultaneously
  - one approach may work better than the others

Multiple Inputs: Knapsack

> We have seen a similar example already: the Knapsack problem

> Inputs are:
  - list of items, 1 .. n
  - price limit W

not a list, but still a separate input

> Solved every sub-problem of the form 1 .. j and V with j ≤ n and V ≤ W
  - total of n(W+1) sub-problem

trying to use last pound may not work, but trying last item works wel

#	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Outline for Today

> Multiple Inputs Generally
> Longest Common Subsequence
> Edit Distance
> Pattern Matching

Longest Common Subsequence

> **Definition**: A subsequence of a list a_1, ..., a_n is a list c_1, ..., c_k, where each c_i is from the first list and they appear in the *same order*.

> Note that the indices need **not** be **contiguous**:
  – sub-sequences not ranges / sub-arrays

> E.g., if A = [3, 8, -5, 0, 23, 4],
then B = [8, 0, 23] is a subsequence (not a subarray)
C = [3, 8, 5] is **not** a subsequence (no 5 in A)
D = [3, 4, 8] is **not** a subsequence (8 before 4 in A)

Longest Common Subsequence

- **Problem**: Given two lists, $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, find the longest subsequences of the two lists that are identical.
  - subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of $a_1, a_2, \ldots, a_n$ and subsequence $b_{i_1}, b_{i_2}, \ldots, b_{i_k}$ of $b_1, b_2, \ldots, b_m$ with $a_{i_1} = b_{i_1}, a_{i_2} = b_{i_2}, \ldots, a_{i_k} = b_{i_k}$

- Example:
  - A = [1, 2, 1, 5, 4, 3]
  - B = [2, 1, 3, 2, 1, 4]
  - [1, 2, 1, 4] is the longest common subsequence

Longest Common Subsequence

- Brute force would take $\Omega(4^{\min(n,m)})$ time
  - try all $\geq 2^{\min(n,m)}$ subsets of $a_1, \ldots, a_n$ with length at most min(n,m)
  - try all $\geq 2^{\min(n,m)}$ subsets of $b_1, \ldots, b_m$ with length at most min(n,m)
  - return the longest match found

Longest Common Subsequence

- Brute force would take $O(4^{\min(n,m)})$ time
- Apply dynamic programming...
- **Q**: How does the opt solution use the last elements (a_n and b_m)?
  - could use just a_n, just b_m, both, or neither

Longest Common Subsequence

- Apply dynamic programming...

- **Q**: How does the opt solution use the last elements (a_n and b_m)?
  - uses neither: same as opt on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{m-1}$
  - uses only a_n: same as opt on $a_1, \ldots, a_n$ and $b_1, \ldots, b_{m-1}$ ← b_m not needed
  - uses only b_m: same as opt on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_m$ ← a_n not needed
  - uses both...
    - then we must have $a_n = b_m$
    - rest must be opt on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{m-1}$
    - opt value = 1 + opt value on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{m-1}$
      - each common subsequence on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{m-1}$ becomes 1 longer by adding a_n and b_m, so opt must use longest of those

Longest common subsequence

- Apply dynamic programming...
  1. Can find opt value for 1, ..., n (a) and 1, ..., m (b) using
     (i) opt value for 1, ..., n-1 and 1, ..., m
     (ii) opt value for 1, ..., n and 1, ..., m-1
     (iii) opt value for 1, ..., n-1 and 1, ..., m-1
  2. Need opt values sub-problems on 1, ..., i (a) and1, .., j (b) with i ≤ n and j ≤ m

- (n+1)(m+1) problem to solve
  - let i or j be zero (empty prefixes)

Longest common subsequence

- Apply dynamic programming...
  1. Can find opt value for 1, ..., n (a) and 1, ..., m (b) using prefixes of each.
  2. Need opt values sub-problems on 1, ..., i (a) and1, .., j (b) with i ≤ n and j ≤ m
  3. Solve each of these starting with i=0 or j=0
     - opt value = 0 if i = 0 or j = 0
     - opt value for 1, ..., i and 1, ..., j =
       max(opt value for 1, ..., i-1 and 1, ..., j,
       opt value for 1, ..., i and 1, ..., j-1,
       (opt value for 1, ..., i-1 and 1, ..., j-1) + (1 if $a_i = b_j$ else 0))
- O(1) per table entry, so O(nm) time all together

Outline for Today

> Multiple Inputs Generally
> Longest Common Subsequence
> Edit Distance
> Pattern Matching

Edit Distance

> **Problem**: Given two lists, $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, find the minimum cost way to transform a into b using three operations:
>
> 1. Change element v to element w at cost $\alpha_{v,w}$
> 2. Insert element v at cost β_v
> 3. Delete element v at cost δ_v

Edit Distance Example

Edit distance between these two strings (DNA):

C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	T	A	C	A	T

- Mismatch at all the blue locations
- Cost of those mismatches is $\alpha_{C,T} + \alpha_{G,T} + \alpha_{A,G} + 2\,\alpha_{A,C}$

Edit Distance Example

Edit distance between these two strings (DNA):

–	C	T	G	A	C	C	T	A	C	C	T
C	C	T	G	A	C	–	T	A	C	A	T

> Alternatively:
  - insert ”C” at the beginning (top “–”)
  - delete “C” in the middle (across from bottom “–”)
  - cost is $\beta_C + \delta_C + \alpha_{A,C}$

Edit Distance Applications

- Computational biology (“sequence alignment”)
  - measures similarity between DNA (or RNA or proteins)
  - cost of insert / delete / change based on likelihood of mutations
- Spell checkers
  - cost of insert / delete / change based on likelihood of those mistakes
- Diff tool
- Speech recognition

Edit Distance Applications

- Longest common subsequence:
  - insertion and deletion cost 1, changes costs ∞
  - for any common subsequence of length k, can first into second by:
    - deleting n – k other elements from a
    - inserting m – k other elements into b
- Example:
  - A = [1, 2, 1, 5, 4, 3]
  - B = [2, 1, 3, 2, 1, 4]
  - delete 5 & 3 from A to get [1, 2, 1, 4] (common subsequence)
  - insert 2 & 3 to this to get B

Edit Distance Applications

- Longest common subsequence:
  - insertion and deletion cost 1, changes costs ∞
  - for any common subsequence of length k, can first into second by:
    - deleting n – k other elements from a
    - inserting m – k other elements into b
    - total cost is n + m – 2k
  - since n + m is constant, minimizing n + m – 2k is maximizing over k

- Edit distance generalizes longest common subsequence
  - another example of robustness to problem changes
  - also suggests previous solution will work here too…

Edit Distance

- Apply dynamic programming...
- **Q**: How does the opt solution *match* the last elements (a_n and b_m)?
  - if $a_n = b_m$: opt value = opt on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{m-1}$
  - if change: opt value = $\alpha_{v,w}$ + opt on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_{m-1}$
  - if insert b_m: opt value = β_v + opt on $a_1, \ldots, a_n$ and $b_1, \ldots, b_{m-1}$
  - if delete a_n: opt value = δ_v + opt on $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_m$

Edit Distance

- Apply dynamic programming...
  1. Can find opt value for 1, ..., n (a) and 1, ..., m (b) using prefixes of each.
  2. Need opt values sub-problems on 1, ..., i (a) and1, .., j (b) with $i \leq n$ and $j \leq m$
  3. Solve each of these starting with i=0 or j=0
     - if i = 0, then opt value = $\beta_{b1} + \ldots + \beta_{bj}$
     - if j = 0, then opt value = $\delta_{a1} + \ldots + \delta_{ai}$
     - opt value for 1, ..., i and 1, ..., j =

       max($\alpha_{v,w}$ + opt value for 1, ..., i-1 and 1, ..., j-1,
       β_v + opt value for 1, ..., i and 1, ..., j-1,
       δ_v + opt value for 1, ..., i-1 and 1, ..., j),

       where $v = a_i$ and $w = b_j$

← set $\alpha_{v,w} = 0$ when $v = w$

Edit Distance

- Running time is O(nm) as before
- Very easy to implement
  - about 10 lines of code (see the textbook)
- Easily implemented in Excel
  - filling in a 2D table
  - each value is a minimum of 4 others

Foreword: Edit Distance Memory Reqs

> In computation biology, n and m could be very large…
  - with n = m = 100k, nm = 10b
  - running time is fine since modern machines perform billions of ops per sec
  - memory use of 10GB (assuming 1B per entry) is (just) possible

> With n = m = 1,000,000 though:
  - running time is okay: 1000B operations in minutes
  - memory use of 1TB is not reasonable
    > could use disk space, but time would increase by factor of ~1k

> More on this next time…

Outline for Today

- Multiple Inputs Generally
- Longest Common Subsequence
- Edit Distance
- Pattern Matching ⬅

Pattern Matching

> **Problem**: Given a content string $a_1, \ldots, a_n$ and a pattern $p_1, \ldots, p_m$, find the longest substring of the content that matches the pattern according to the following rules:
  - '?' in the pattern matches any single character of content
  - '*' in the pattern matches any substring (including an empty one)
  - any other letter in the pattern matches only the *same* letter of content

Pattern Matching Examples

> Content "abcba"
> Pattern "a*b"
> – longest match is prefix "abcb"

> Content "abcba"
> Pattern "b?b"
> – longest match is "bcb"

> Content "abcba"
> Pattern "b??a"
> – longest match is suffix "bcba"

Pattern Matching Applications

> Common feature of editors and IDEs

> Many also support regular expression matching
  - RE matching is part of most standard libraries
  - more on that later…

Pattern Matching

- Apply dynamic programming...

- Like max sub-array sum, it will be helpful to change the problem: find the longest match **ending at a_n**
  - apply DP to the original problem and you will find you need to solve these
  - but these are also sufficient to solve the whole problem
    - every match ends somewhere
    - longest over the longest ending at $a_1, \ldots, a_n$ is the longest overall

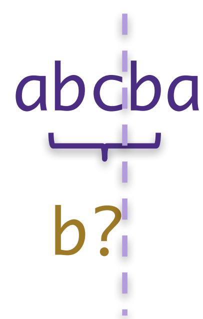
Pattern Matching

- Apply dynamic programming...
  - can consider how either a_n or p_m (or both) is used by the longest match
  - turns out to be easiest to think about how p_m is used
    - in practice, just try all and see what works

- **Q**: How does the longest match use p_m?
- Depends on what p_m is
  - p_m is a letter
  - p_m is a '?'
  - p_m is a '*'

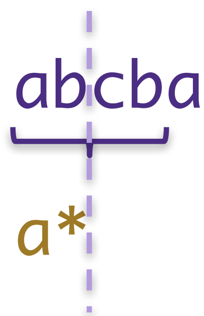
Pattern Matching

- **Q**: How does the longest match use p_m? Depends on what p_m is...
  - p_m is a '?'
    - if '?' matches a_m, then $p_1, \ldots, p_{m-1}$ matches $a_1, \ldots, a_{n-1}$
    - longest match is the longest match of 1 .. n-1 with 1 .. m-1



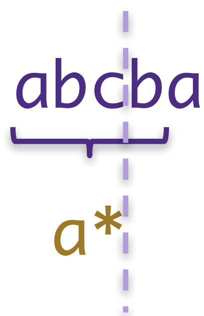
Pattern Matching

- **Q**: How does the longest match use p_m? Depends on what p_m is...
  - p_m is a '?'
    - if '?' matches a_m, then $p_1, \ldots, p_{m-1}$ matches $a_1, \ldots, a_{n-1}$
    - longest match is the longest match of 1 .. n-1 with 1 .. m-1
  - p_m is a '*'
    - if '*' matches a_m, then $a_1, \ldots, a_{n-1}$ **either** matches $p_1, \ldots, p_{m-1}$ **or** $p_1, \ldots, p_m$



Pattern Matching

- **Q**: How does the longest match use p_m? Depends on what p_m is...
  - p_m is a '?'
    - if '?' matches a_m, then $p_1, \ldots, p_{m-1}$ matches $a_1, \ldots, a_{n-1}$
    - longest match is the longest match of 1 .. n-1 with 1 .. m-1
  - p_m is a '*'
    - if '*' matches a_m, then $a_1, \ldots, a_{n-1}$ **either** matches $p_1, \ldots, p_{m-1}$ **or** $p_1, \ldots, p_m$



Pattern Matching

- **Q**: How does the longest match use p_m? Depends on what p_m is...
  - p_m is a '?'
    - if '?' matches a_m, then $p_1, \ldots, p_{m-1}$ matches $a_1, \ldots, a_{n-1}$
    - longest match is the longest match of 1 .. n-1 with 1 .. m-1
  - p_m is a '*'
    - if '*' matches a_m, then $a_1, \ldots, a_{n-1}$ **either** matches $p_1, \ldots, p_{m-1}$ **or** $p_1, \ldots, p_m$
    - longest match is the longer of match of 1 .. n-1 with 1 .. m-1 and 1 .. n-1 with 1 .. m

actually, this has a problem...

it does not allow the '*' to match *nothing*

Pattern Matching

- **Q**: How does the longest match use p_m? Depends on what p_m is...
  - p_m is a '?'
    - if '?' matches a_m, then $p_1, \ldots, p_{m-1}$ matches $a_1, \ldots, a_{n-1}$
    - longest match is the longest match of 1 .. n-1 with 1 .. m-1
  - p_m is a '*'
    - then **either** $a_1, \ldots, a_{n-1}$ matches $p_1, \ldots, p_m$ **or** $a_1, \ldots, a_n$ matches $p_1, \ldots, p_{m-1}$
    - longest match is the longer of match of 1 .. n-1 with 1 .. m and 1 .. n with 1 .. m-1

either match a_n with same pattern
or '*' matches nothing
(can still match multiple characters)

Pattern Matching

- **Q**: How does the longest match use p_m? Depends on what p_m is...
  - p_m is a '?'
    - if '?' matches a_m, then $p_1, \ldots, p_{m-1}$ matches $a_1, \ldots, a_{n-1}$
    - longest match is the longest match of 1 .. n-1 with 1 .. m-1
  - p_m is a '*'
    - then either $a_1, \ldots, a_{n-1}$ matches $p_1, \ldots, p_m$ or $a_1, \ldots, a_n$ matches $p_1, \ldots, p_{m-1}$
    - longest match is the longer of match of 1 .. n-1 with 1 .. m and 1 .. n with 1 .. m-1
  - p_m is a letter
    - if p_m matches a_m, then
longest match is the longest match of 1 .. n-1 with 1 .. m-1
    - if p_m does not match a_m, then there is no match

Pattern Matching

- Apply dynamic programming…
  1. Can find longest match for 1 .. n (a) and 1 .. m (p) using prefixes of each
  2. Need longest match on 1, …, i (a) and1, .., j (p) with i ≤ n and j ≤ m
  3. Solve each of these starting with i=0
     - longest match starts at i+1 if j=0
       - that indicates the range i+1 .. i, which is *empty*
     - longest match starts at infinity if i=0 (and j > 0)
       - that indicates *no range*
     - longest match for 1 .. i and 1 .. j (i > 0 and j > 0) starts at min of four cases on previous slide
       - (if/then's are better written as code… still very short)
       - chose infinity for no range so min will *never* choose it if a match exists

Pattern Matching

- Apply dynamic programming...
  1. Can find longest match for 1 .. n (a) and 1 .. m (p) using prefixes of each
  2. Need longest match on 1, ..., i (a) and1, .., j (p) with $i \leq n$ and $j \leq m$
  3. Solve each of these starting with i=0

- (n+1)(m+1) entries in table, and O(1) time per entry,
  so total running time is O(nm)
  - in practice, n >> m (say, $m \leq 100$), so this is O(n)

- Only needs O(m) memory
  - only need column for i-1 to compute i, so just keep prev column
  - this is why we started with i=0 rather than j=0

Regular Expression Matching (out of scope)

> Regular expressions greatly generalize these simple patterns

> However, the matching algorithm is largely unchanged
  - prefixes of the pattern are replaced with states of the NFSM
  - for our simple patterns, this produces the same result because states of the equivalent NFSM are in 1–to–1 correspondence with prefixes
  - for more general patterns, that is longer the case, so it becomes necessary to determine the NFSM states