CSE 417
Divide & Conquer (pt 4)
More Examples

UNIVERSITY of WASHINGTON

Reminders

- **HW3 due Wednesday**
  - get started right away on understanding the algorithm
  - will be quick to do the drawing *once you understand it fully*

- **HW2 postscript**
  - good resume material
  - saw the impact of regularization
    - penalty improves prediction accuracy (ex. of Occam's razor)
  - esp. useful for model selection on "small data" problems
  - always keep an eye out for applications of binary search

Divide & Conquer Review

- Apply the steps:
  1. Divide the input data into 2+ parts
  2. Recursively solve the problem on each part
  3. Combine the sub-problem solutions into a problem solution
- Key questions:
  1. Can you solve the problem by combining solutions from sub-problems?
  2. Is that easier than solving it directly?
- Use master theorem to calculate the running time

Review from last Time

- **Counting Inversions**
  - combine: add inversions (i, j) with i in first half and j in second half
  - optimized by using binary search ~> faster than $O(n^{1+\varepsilon})$ for any $\varepsilon > 0$
    - eventually realized sequence of linear searches is O(n) ~> so O(n log n) in total
- **Voronoi diagrams**
- **Closest Pair of Points**
  - optimize by comparing (p,q) if $|p.x - q.x| < d$ and $|p.y - q.y| < d$
  - careful analysis reveals this is O(n) *worst case* ~> O(n log n) overall
    - would have seen it was fast in practice anyway

Outline for Today

> Maximum sub-array sum
> Maximum single-sell profit
> Intersecting horz & vert segments

Maximum Sub-array Sum

- Famous interview question from the 1980–90s
  - Jon Bentley wrote about the problem in CACM 1984
  - no longer used, I think, since it's too well known
- Good algorithms question: has multiple solutions
  - the best solution uses dynamic programming
  - can also be solved with divide & conquer (today)
- HW4 is similar to this problem

Maximum Sub-array Sum

> **Problem**: Given an array A of integers,
> find max of A[i] + … + A[j-1] over all $0 \le i \le j \le n$
  - we allow i = j so that the sub-array A[i:j-1] can be empty
  - note that A[i]'s can be **negative**

> Example:

```
A = [31, -41, 59, 26, -53, 58, 97, -93, -23, 84]
```

> Maximum sum is A[2] + … A[6] = 59 + … + 97 = 187
  - includes a negative number, -53

Maximum Sub-array Sum

- **Problem**: Given an array A of integers,
  find max of A[i] + … + A[j-1] over all 0 ≤ i ≤ j ≤ n
  - we allow i = j so that the sub-array A[i:j-1] can be empty
- Brute force solution 1:
  - for every i = 0 .. n, for every j = i .. n, compute A[i] + … + A[j-1]
    - take the maximum of all these
  - takes $\Theta(n^3)$ time

Maximum Sub-array Sum

- **Problem**: Given an array A of integers,
  find max of A[i] + … + A[j-1] over all $0 \le i \le j \le n$
  - we allow i = j so that the sub-array A[i:j-1] can be empty
- Brute force solution 1:
  - takes $\Theta(n^3)$ time
- Brute force solution 2:
  - compute B[i] = A[i] + A[i+1] + … + A[n-1] for all i in $\Theta(n)$ time
  - for every i = 0 .. n, for every j = i .. n, compute B[i] – B[j]
    - take the maximum of all these

Maximum Sub-array Sum

- **Problem**: Given an array A of integers, find max of A[i] + ... + A[j-1] over all $0 \le i \le j \le n$
  - we allow i = j so that the sub-array A[i:j-1] can be empty
- Brute force solution 1:
  - takes $\Theta(n^3)$ time
- Brute force solution 2:
  - for every i = 0 .. n, for every j = i .. n, compute B[i] – B[j]
    - take the maximum of all these
  - takes $\Theta(n^2)$ time

Maximum Sub-array Sum

> Apply divide & conquer...

1. Divide A into halves, A[0..n/2-1] and A[n/2..n-1]

2. Recursively solve the sub-problem on each half
   - sums that start & end on one side

3. Combine by considering sums starting on left, ending on right
   - max (A[i] + … + A[n/2-1]) + (A[n/2] + … + A[j-1])

Maximum Sub-array Sum

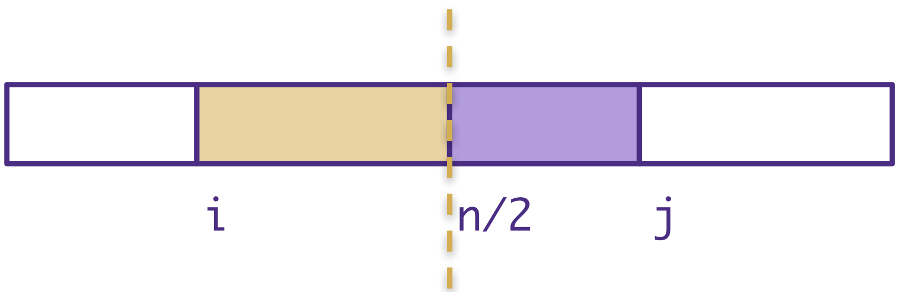
- Apply divide & conquer...

  3. Combine by considering sums starting on left, ending on right

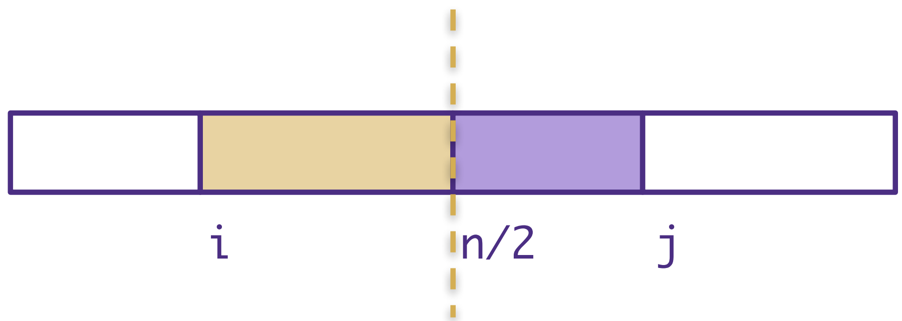
     - maximize A[i] + ... + A[j-1]

Maximum Sub-array Sum

> max A[i] + … + A[j-1]
  = max A[i] + … + A[n/2-1] + A[n/2] + … + A[j-1]
  – can always do this since i ≤ n/2 ≤ j
  – splits sum into two parts
    > sum of suffix of first half
    > sum of prefix of second half



Maximum Sub-array Sum



- > Choice of i only affects first part and
  Choice of j only affects second part
  - – can maximize the two independently

- > max A[i] + … + A[j-1]
  = (max A[i] + … + A[n/2-1]) + (max A[n/2] + … + A[j-1])

Maximum Sub-array Sum

> Apply divide & conquer…

3. Combine by considering sums starting on left, ending on right

> $\max\ (A[i] + \ldots + A[n/2-1]) + (A[n/2] + \ldots + A[j-1])$
> $= (\max\ A[i] + \ldots + A[n/2-1]) + (\max\ A[n/2] + \ldots + A[j-1])$

> that equation is the key insight of the algorithm!
 - max is achieved by **separately** maximizing each half
 - similar ideas in dynamic programming (also greedy)

Maximum Sub-array Sum

> Apply divide & conquer...

3. Combine by considering sums starting on left, ending on right
   - > max (A[i] + … + A[n/2-1]) + (A[n/2] + … + A[j-1])
     = (max A[i] + … + A[n/2-1]) + (max A[n/2] + … + A[j-1])
   - > compute sums A[i] + … + A[n/2-1] for each i = 0 .. n/2-1
     - takes $\Theta(n)$ time
     - take the maximum of these
   - > compute sums A[n/2] + … + A[j-1] for each j = n/2 .. n-1
     - takes $\Theta(n)$ time
   - > take sum of two maximums

Maximum Sub-array Sum

- Apply divide & conquer...

  1. Divide A into halves, A[0..n/2-1] and A[n/2..n-1]
  2. Recursively solve the sub-problem on each half
     - sums that start & end on one side
  3. Combine by considering sums starting on left, ending on right
     - find max sum crossing divide with scan left & scan right

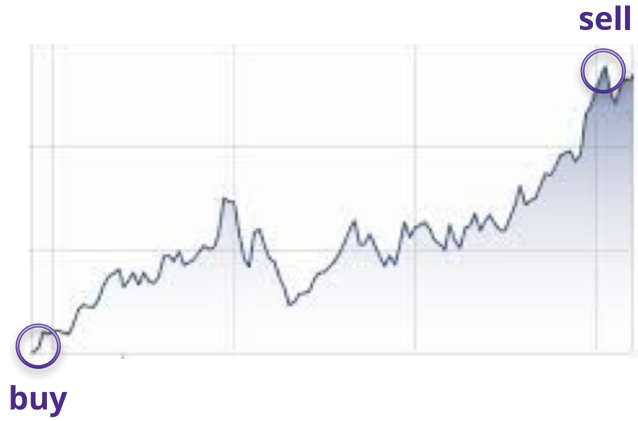
- Divide + combine in $\Theta(n)$, so $\Theta(n \log n)$ by master thm

Outline for Today

> Maximum sub-array sum
> Maximum single-sell profit
> Intersecting horz & vert segments

Maximum Single-Sell Profit

> **Problem**: Given a list of prices on each day, find the days on which to buy & sell that would have achieved max profit
  - maximize (sell price – buy price)



Maximum Single-Sell Profit

> **Issue**: actually want to minimize percentage increase
  – (i.e., increase per dollar bought)
  – can spend whatever amount you want on the stock (simplification)

> max $(\text{price}_{\text{sell}} - \text{price}_{\text{buy}}) / \text{price}_{\text{buy}}$

$= \max \text{price}_{\text{sell}} / \text{price}_{\text{buy}} - 1$

$= \max \text{price}_{\text{sell}} / \text{price}_{\text{buy}}$

$= \max \log(\text{price}_{\text{sell}} / \text{price}_{\text{buy}})$

$= \max \log(\text{price}_{\text{sell}}) - \log(\text{price}_{\text{buy}})$

  – *original problem with log prices instead*

log is monotonically increasing,
so it does not change order,
so it does not change maximum

Maximum Single-Sell Profit

> Apply divide & conquer

1. Divide prices into first half and second half
2. Recursively solve each sub-problem
   > gives best with both buy & sell on same half
3. Combine by considering best buy on one half and sell on other half
   > only need to consider buying in first half and selling in second half because we cannot buy after we sell
   - (actually, you can… it's called short selling, but that's disallowed here)

Maximum Single-Sell Profit

> Apply divide & conquer

1. Combine by considering best (buy in first half, sell in second half)
   - > Q: what is the best price to buy at in first half?
   - > A: minimum price
   - > Q: what is the best price to sell at in second half?
   - > A: maximum price

Maximum Single-Sell Profit

> Apply divide & conquer

1. Divide prices into first half and second half

2. Recursively solve each sub-problem
   > gives best with both buy & sell on same half

3. Combine by considering best buy on one half and sell on other half
   > find (min(first half), max(second half)) in O(n) time

> Running time is O(n log n) by master theorem

Maximum Single-Sell Profit

- > **Q**: Is this *really* easier to solve by divide and conquer?
- > **A**: No!

- > Linear-time single-pass (right to left) algorithm:
  - keep track of max profit seen so far
  - keep track of the maximum price seen so far
    - > best available price to sell in the future
  - if current price – max price > max profit:
    - > make this the max price

- > This is O(n) — faster than divide & conquer solution

Maximum Single-Sell Profit

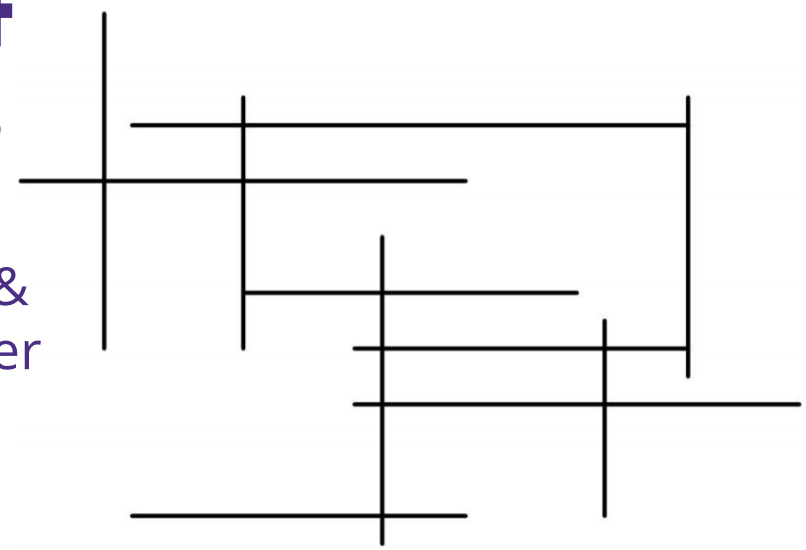
- **Q**: Is this *really* easier to solve by divide and conquer?
- **A**: No!

- Divide & Conquer may still help us **find** the solution
- BUT ask yourself when you're done whether it's really needed

Outline for Today

> Maximum sub-array sum
> Maximum single-sell profit
> Intersecting horz & vert segments

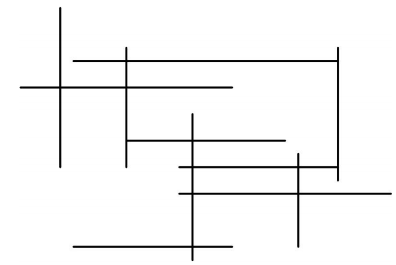
Count Intersections of Horz & Vert Segments



- > **Problem**: given a set of horizontal & vertical segments, count the number of points where they intersect
- > Brute force solution:
  - – check every pair to see if they intersect
  - – $\Theta(n^2)$ pairs to check if n/2 vertical and n/2 horizontal segments

Picture from www.inrg.csie.ntu.edu.tw/algorithm2014/course/Divide%20&%20Conquer.pdf

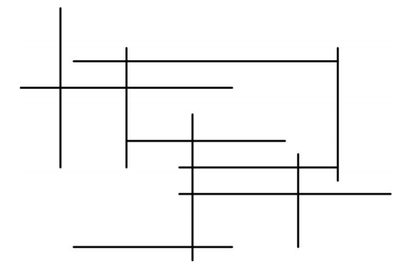
Count Intersections of Horizontal & Vertical Segments



> Apply divide & conquer…

1. Divide horizontally into two halves
   - vertical segments end up on one side
   - but horizontal segments could cross the dividing line
   - *leave those out*
2. Recursively solve each sub-problem
3. Combine by adding intersections with missing ones
   - every other intersection is accounted for
   - those completely on left and right cannot intersect

Count Intersections of Horizontal & Vertical Segments

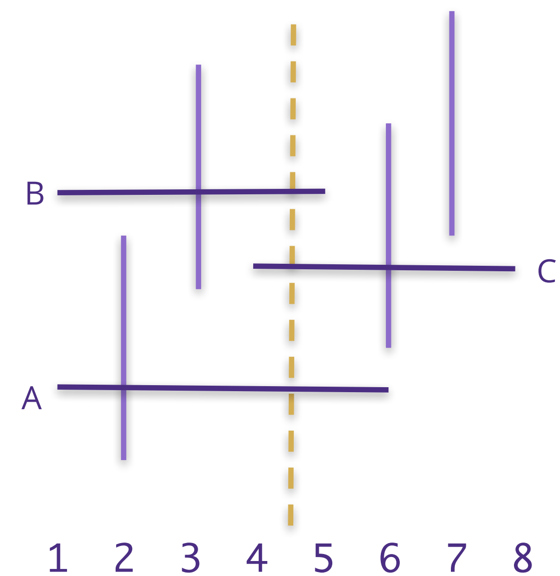


- **Sub-problem**: find intersections with segments left out
  - these are horizontal segments, so they intersect with vertical ones
  - have a list of all the left out horizontal segments

- **Idea**: sort vertical & left out horizontal segments together
  - horizontal segments appear **twice** (once for each endpoint)
  - can do the sort before-hand…
    - only adds O(n log n) to total running time

Count Intersections of Horz & Vert Segments

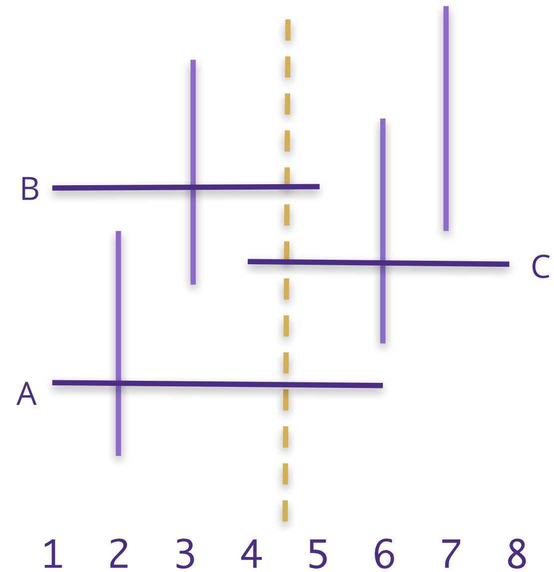
- **Idea**: sort vert & left out vert segments
  - horizontal segments appear **twice**
  - can do the sort before-hand...
    - only adds O(n log n) to total running time



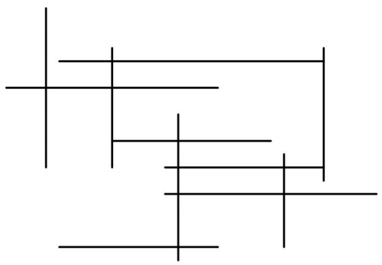
- Example: [1_A, 1_B, 2, 3, 4_C, 5_B, 6, 6_A, 7, 8_C]
  - subscript indicates horizontal segment
  - (in Java, this would be a list of objects)
- Approach: scan from right & left to dividing line
  - track horizontal segments & look for intersections with vertical

Count Intersections of Horz & Vert Segments

- > **Idea**: sort vert & left out vert segments
  - – horizontal segments appear **twice**
- > Example: [1_A, 1_B, 2, 3, 4_C, 5_B, 6, 6_A, 7, 8_C]
  - – subscript indicates horizontal segment
- > Scan right to dividing line
  - – keep an **AVL tree** of horizontal segments seen so far
  - – for each vertical segment:
    - > use range search on AVL tree to find intersecting horz segments
    - > takes O(log n + #intersections) time

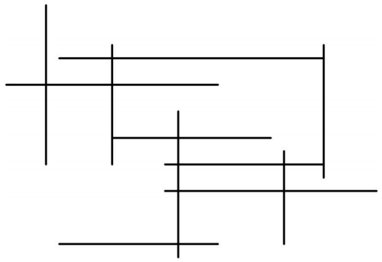


Count Intersections of Horizontal & Vertical Segments



- Apply divide & conquer…
  1. Divide horizontally into two halves
     - leaving out horizontal segments that span two halves
  2. Recursively solve each sub-problem
  3. Combine by adding intersections with missing ones
     - O(n log n + #intersections) scan to left & right to count these
- Running time is $O(n^{1+\varepsilon} + \#\text{intersections})$ by master

Count Intersections of Horizontal & Vertical Segments



- Q: Is this *really* easier to solve by divide and conquer?

- A: No!
  - could just do this linear scan the entire way across
  - running time is O(n log n + #intersections)

- Many faster algorithms for this problem...
  - from Bentley, Tarjan, etc.