# CSE 417
# Divide & Conquer (pt 1)

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# Reminders

> **HW2 is posted: due in one week**
  - start early!
  - added hint on problem 1
    - (use domain knowledge to limit ternary search range)

# Outline for Today

> Divide & Conquer definition
> Some familiar problems
> Master Theorem

# Divide & Conquer

Algorithmic approach:

1. **Divide** the input data into 2+ parts
2. **Recurs**ively solve the problem on each part
   - i.e., solve the *same problem* on each part
3. **Combine** those solutions to solve the original problem

# Sub-problems

- Divide & conquer uses the solutions to **sub-problems** to solve the given problem
  - used term “sub-problem” before as any other problem I solve as a step toward solving the larger problem
    - e.g., to find the N-th special composite, we solve the sub-problem of finding the next largest such number given all the previous ones
  - here, sub-problem is another instance of the same type of problem
    - with N-th special composite, we changed the type of problem

- We will see the same idea with dynamic programming
  - but has a more general use of sub-problems

# Outline for Today

> Divide & Conquer definition
> Some familiar problems
> Master Theorem

# Merge Sort

1. Divide the array A into two halves: A[0..n/2-1] and A[n/2..n-1]
   - where n is the length of A
2. Recursively sort the two halves
3. Combine two sorted arrays into a single sorted array
   - merging sorted arrays is *easier* than sorting
   - use a “two finger” algorithm
     - note that this needs attention to detail

# Merge Sort: Divide

- Divide takes no work at all
  - we don't even copy the arrays
- To do that, we need to *generalize* the inputs:

```
void mergeSort(int[] A, int start, int end)
```

- This issue will come up again with dynamic programming
  - but there it will be less obvious
  - keep this example in mind for later...

# Merge Sort: Combine

> merge with a "two finger" algorithm
  - one finger for the next largest element of each sub-array
  - copy smaller to the output and move that finger forward

> note that this needs attention to detail...

# Merge Sort: Combine

```
/** Make B[i..k-1] the sorted merge of A[i..j-1] and A[j..k-1]. */
void merge(int[] A, int i, int j, int k, int[] B) {
  int a = i, b = j, c = i;

  // Inv: B[i..c-1] is merge of A[i..a-1] and A[j..b-1]
  while (c < k) {
    if (A[a] < A[b]) B[c++] = A[a++];
    else             B[c++] = A[b++];
  }
}
```

**What's wrong?**

# Merge Sort: Combine

```
/** Make B[i..k-1] the sorted merge of A[i..j-1] and A[j..k-1]. */
void merge(int[] A, int i, int j, int k, int[] B) {
  int a = i, b = j, c = i;

  // Inv: B[i..c-1] is merge of A[i..a-1] and A[j..b-1]
  while (a < j && b < k) {
    if (A[a] < A[b]) B[c++] = A[a++];
    else             B[c++] = A[b++];
  }
}
```

**What’s still wrong?**

# Merge Sort: Combine

```
/** Make B[i..k-1] the sorted merge of A[i..j-1] and A[j..k-1]. */
void merge(int[] A, int i, int j, int k, int[] B) {
  int a = i, b = j, c = i;

  // Inv: B[i..c-1] is merge of A[i..a-1] and A[j..b-1]
  while (a < j && b < k) {
    if (A[a] < A[b]) B[c++] = A[a++];
    else             B[c++] = A[b++];
  }

  while (a < j) B[c++] = A[a++];
  while (b < k) B[c++] = A[b++];
}
```

# Merge Sort: Combine

- merge with a "two finger" algorithm
  - one finger for the next largest element of each sub-array
  - copy smaller to the output and move that finger forward
- note that this needs attention to detail
  - easily to forget array out of bounds cases & left over elements
  - you'll spot the error when it crashes, but that's no help for interviews...
- Runs in O(n) time (where n = k – i)
  - every iteration copies one value to B
  - only k – i values to copy

# Merge Sort: running time

- Total time is T(n) where

  $T(1) = O(1)$

  $T(n) = 2\,T(n / 2) + O(n)$ for $n > 1$

- Takes O(1) time to sort 1 element
  - just return
- Time to sort n elements is the time for
  2 recursive calls on half the data + O(n) merge
  - for now, ignore issues about rounding n / 2 to an integer

# Merge Sort: running time

$T(1) = D$

$T(n) = 2\,T(n / 2) + C\,n \qquad\qquad \text{for } n > 1$

> Can solve by repeated substitution…

$$
\begin{aligned}
T(n) &= 2\,(2\,T(n / 4) + C\,(n/2)) + C\,n \\
&= 4\,T(n / 4) + 2\,C\,(n/2) + C\,n \\
&= 4\,T(n / 4) + C\,n + C\,n \\
&= 4\,T(n / 4) + 2C\,n
\end{aligned}
$$

# Merge Sort: running time

$T(1) = D$

$T(n) = 2\,T(n / 2) + C\,n$ for $n > 1$

- Can solve by repeated substitution…

$$
\begin{aligned}
T(n) &= 4\,T(n / 4) + 2C\,n \\
&= 4\,(2\,T(n / 8) + C\,(n / 4)) + 2C\,n \\
&= 8\,T(n / 8) + 4\,C\,(n / 4) + 2C\,n \\
&= 8\,T(n / 8) + 3C\,n
\end{aligned}
$$

# Merge Sort: running time

$T(1) = D$

$T(n) = 2\ T(n / 2) + C\ n$ for $n > 1$

> Can solve by repeated substitution...

$$
\begin{aligned}
T(n) &= 8\ T(n / 8) + 3C\ n \\
&= \ldots \\
&= 2^k\ T(n / 2^k) + kC\ n
\end{aligned}
$$

# Merge Sort: running time

$$T(1) = D$$
$$T(n) = 2\,T(n / 2) + C\,n \qquad \text{for } n > 1$$

> Can solve by repeated substitution…

$$T(n) = 2^k\,T(n / 2^k) + kC\,n$$

> When $k = \log_2 n$, we have $2^k = 2^{\lg n} = n$, so …

$$\begin{aligned} T(n) &= n\,T(n / n) + (\lg n)\,C\,n \\ &= n\,T(1) + C\,n \lg n \end{aligned}$$

# Merge Sort: running time

$T(1) = D$

$T(n) = 2\ T(n / 2) + C\ n$ for $n > 1$

> When $k = \log_2 n$, we have $2^k = 2^{\lg n} = n$, so ...

$$
\begin{aligned}
T(n) &= n\ T(n / n) + (\lg n)\ C\ n \\
&= n\ T(1) + C\ n \lg n \\
&= D\ n + C\ n \lg n \\
&= O(n \log n)
\end{aligned}
$$

# Merge Sort 2: sort a linked list

- Almost certainly the best algorithm for linked lists
  - unlike array version, does not require any extra memory
    - still needs extra space, but…
    - gets by with the space in the pointers of the linked list nodes
  - with arrays, quick sort was at one time considered fastest
    - merge sort is probably more commonly used there also now
    - changes in processor architecture had an impact

- Use the same divide & conquer approach…
  - split, recurse, merge

# Merge Sort 2: Divide

> How do you split a linked list in two halves?

1. Find an element in the middle and disconnect the lists there
2. Put the even elements in one list and the odds in another

> Let’s look at the second one…

# Merge Sort 2: Divide

```
/** Put half of A’s elements into B and half into C */
<T> void split(LinkedList<T> A, List<T> B, List<T> C) {
  while (A.size() > 0) {
    B.add(A.removeFirst());
    C.add(A.removeFirst());
  }
}
```

**What’s wrong?**

# Merge Sort 2: Divide

```
/* Put half of A's elements into B and half into C */
<T> void split(LinkedList<T> A, List<T> B, List<T> C) {
  while (A.size() > 0) {
    if (A.size() % 2 == 0)
      B.add(A.removeFirst());
    else
      C.add(A.removeFirst());
  }
}
```

# Merge Sort 2: Combine

- Merge two sorted linked lists into one with the same idea
  - “two finger” algorithm
- one finger points to next node in each list
  - smallest value not yet merged into the result
- merged list is initially empty
  - add smallest node to the end

# Merge Sort 2: running time

> Now, the divide is not O(1), it is O(n)

> Combine is still an O(n) merge

> Total time is T(n) where

$$T(1) = O(1)$$

$$T(n) = 2\,T(n / 2) + O(n) \qquad \text{for } n > 1$$

> Same formula, so same running time as before O(n log n)

# Finding the minimum

> **Problem**: Find the minimum value in A[0..n-1]

1. Divide the array into two halves: A[0..n/2-1] and A[n/2..n-1]
2. Recursively find the minimums: $m_1$ and $m_2$
3. Combine to find the overall minimum: $\min(m_1, m_2)$

> Is this the best way to solve the problem?
  – not really…

# Finding the minimum: running time

> Total time is $T(n)$ where

$$T(1) = O(1)$$
$$T(n) = 2\,T(n/2) + O(1) \qquad \text{for } n > 1$$

# Finding the minimum: running time

$T(1) = D$

$T(n) = 2\ T(n / 2) + C \qquad \text{for } n > 1$

- Solve by repeated substitution…

$$
\begin{aligned}
T(n) &= 2\ (2\ T(n / 4) + C) + C \\
&= 4\ T(n / 4) + C\ (2 + 1) \\
&= 4\ (2\ T(n / 8) + C) + C\ (2 + 1) \\
&= 8\ T(n / 8) + C\ (4 + 2 + 1)
\end{aligned}
$$

# Finding the minimum: running time

$T(1) = D$

$T(n) = 2\ T(n / 2) + C \qquad \text{for } n > 1$

- Solve by repeated substitution…

$T(n) = 2^k\ T(n / 2^k) + C\ (2^k + \ldots + 2 + 1)$

- When $k = \log_2 n$, $2^k = n$, so …

$T(n) = n\ T(n / n) + C\ (2^k + \ldots + 2 + 1)$

# Finding the minimum: running time

$T(1) = D$

$T(n) = 2\ T(n / 2) + C$ for $n > 1$

> When $k = \log_2 n$, $2^k = n$, so …

$$T(n) = n\ T(n / n) + C\ 2^k\ (1 + \ldots + 1/2^{k-1} + 1/2^k)$$
$$= n\ T(1) + C\ n\ (1 + 1/2 + \ldots + 1/2^k)$$
$$= D\ n + C\ n\ (1 + 1/2 + \ldots + 1/2^k)$$
$$= D\ n + C\ n\ O(1)$$
$$= O(n)$$

# Outline for Today

> Divide & Conquer definition
> Some familiar problems
> Master Theorem

# Master Theorem

> Solves recurrence relations of the form that arise in Divide & Conquer algorithms:

$T(1) = O(1)$
$T(n) <= a\ T(n/b) + f(n)$

- divide into **a** problems of size **n / b**
- divide + combine takes f(n) time

# Master Theorem

**Theorem**: Let $T(n)$ be bounded as on the previous slide.
Define $C := \log_b a$. Then…

# size 1 problems

time dominated
by all those O(1)'s

- If $f(n) = O(n^{C-\varepsilon})$ for any $\varepsilon > 0$, then $T(n) = \Theta(n^C)$
- If $f(n) = \Theta(n^C)$, then $T(n) = \Theta(n^C \log n)$

time dominated
by divide+combine
of initial problem

- If $f(n) = \Omega(n^{C+\varepsilon})$ for any $\varepsilon > 0$ and …, then $T(n) = \Theta(f(n))$
  - … and f must satisfy $a\, f(n / b) < C\, f(n)$ for some $C < 1$
  - … almost always true for polynomial time algorithms
  - … don't worry about it

# Example: Binary Search

- divide into 1 sub-problem of size n / 2
- so a = 1 and b = 2 and C = $\log_2 1 = 0$

- divide take O(1) time: compare middle element to x
- combine takes no time
  - actually, takes O(1) just for the return statement
- so f(n) = Θ(1)

# Example: Binary Search

> $C = \log_2 1 = 0$
> $f(n) = \Theta(1)$

> Compare $f(n)$ to $n^C = n^0 = 1$...

> $f(n) = \Theta(n^C)$ since both are $\Theta(1)$
> Master theorem says time is $\Theta(n^C \log n) = \Theta(\log n)$

# Example: Binary Search

- f(n) = $\Theta(n^C)$ since both are $\Theta(1)$
- Master theorem says time is $\Theta(n^C \log n) = \Theta(\log n)$

- If we had $f(n) = \Theta(n^{0.5})$,
  then master theorem says time is $\Theta(f(n)) = \Theta(n^{0.5})$
  - time is dominated by the divide + combine of the initial problem

# Example: Merge Sort

- divide into **2** sub-problem of size n / 2
- so a = 2 and b = 2 and C = $\log_2 2$ = **1**

- divide take no time: sub-arrays are already in place
- combine takes O(n) time
  - two-finger pass over the two sorted sub-arrays
- so f(n) = $\Theta$(n)

# Example: Merge Sort

- $C = \log_2 2 = 1$
- $f(n) = \Theta(n)$

- Compare $f(n)$ to $n^C = n^1 = n$...

- $f(n) = \Theta(n^C)$ since both are $\Theta(n)$
- Master theorem says time is $\Theta(n^C \log n) = \Theta(n \log n)$

# Example: Merge Sort

> f(n) = $\Theta(n^c)$ since both are $\Theta(n)$
> Master theorem says time is $\Theta(n^c \log n) = \Theta(n \log n)$

> If we had f(n) = $\Theta(n^{0.5})$,
> then master theorem says time is $\Theta(n^c) = \Theta(n)$
> – time is dominated by all the O(1) works in size 1 problems

# Example: Merge Sort

- $f(n) = \Theta(n^c)$ since both are $\Theta(n)$
- Master theorem says time is $\Theta(n^c \log n) = \Theta(n \log n)$

- If we had $f(n) = \Theta(n^2)$,
  then master theorem says time is $\Theta(n^2)$
  - time is dominated by the divide + combine of initial problem

# Example: Merge Sort

- $f(n) = \Theta(n^c)$ since both are $\Theta(n)$
- Master theorem says time is $\Theta(n^c \log n) = \Theta(n \log n)$

- If we had $f(n) = \Theta(n \log n)$,
  then master theorem says…

# Example: Merge Sort

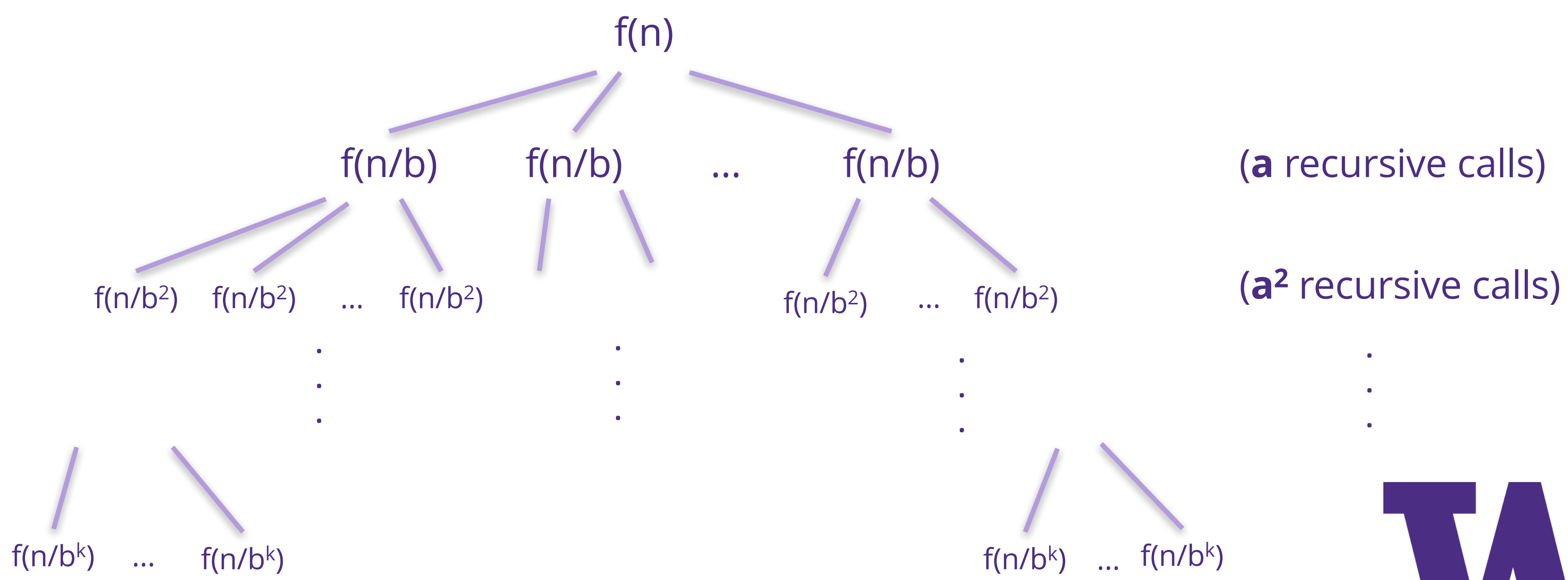
- $f(n) = \Theta(n^C)$ since both are $\Theta(n)$
- Master theorem says time is $\Theta(n^C \log n) = \Theta(n \log n)$

- If we had $f(n) = \Theta(n \log n)$,
  then master theorem says… **nothing**!
  - need $f(n) = \Omega(n^{C + \varepsilon})$ for some $\varepsilon > 0$
  - even $f(n) = \Omega(n^{1.00000001})$ would work but $\Theta(n \log n)$ does not

# Understanding the Master Theorem

- Recursion tree
  - node for each recursive call
- Node records time for the divide + combine of that call
- Total time for any call is the sum of all nodes in the subtree
  - cost of the recursive calls is recorded in those child nodes

# Understanding the Master Theorem



# Understanding the Master Theorem

- Bottom level has $n / b^k = 1$, so $\mathbf{k = \log_b n}$
  - height of the tree is $k + 1$

- Number of leaves is $a^k$
  - this is $a^{\log_b n} = a^{\log_a n \log_b a} = n^{\log_b a} = \mathbf{n^c}$

# Understanding the Master Theorem

> Time for the top call (divide + combine) is $f(n)$
> Time for all the calls in the leaves is $O(1) \times n^C = \Theta(n^C)$

> Master theorem asks us to compare these times
  - when $f(n) \gg \Theta(n^C)$, then the root divide + combine dominates everything
  - when $f(n) \ll \Theta(n^C)$, then the leaf node work dominates everything
  - when $f(n) = \Theta(n^C)$, then all the $\Theta(\log n)$ levels do the same work

# Proof of Master Theorem (out of scope)

> Total time is

$$\Theta(n^{C}) + \sum_{i=0}^{\log_{bn} - 1} a^{i} f(n/bi)$$

- left term is leaves
- right term is all the divide + combines
- (need two terms since T(n) is defined by two formulas)

# Proof of Master Theorem (out of scope)

- **Case 2**: $f(n) = \Theta(n^C)$
- Let $f(n) <= A\ n^C$
  - so A is the hidden constant in the big-O
- Then $a^i f(n / b^i) <= a^i A (n / b^i)^C = A n^C a^i / b^{iC} = A n^C (a / b^C)^i$
  - note that $b^C = b^{\log_b a} = a$
  - so we have $a^i f(n / b^i) <= A n^C (a / a)^i = A n^C$
  - work on each level is $<= A n^C$
  - total work is $<= \Theta(\log n) A n^C = \Theta(f(n) \log n)$

# Proof of Master Theorem (out of scope)

- **Case 1**: $f(n) << \Theta(n^C)$
  - need to show summation is $O(n^C)$
  - then the first term is just as big

- Assume $f(n) <= A\ n^{C-\varepsilon}$ for some $\varepsilon > 0$
- Then $a^i\ f(n / b^i) <= a^i\ A\ (n / b^i)^{C-\varepsilon} = (n / b^i)^{-\varepsilon}\ A\ n^C\ (a / b^C)^i = A\ n^{C-\varepsilon}\ b^{i\varepsilon}$
  - only difference is factor of $(n / b^i)^{-\varepsilon}$
  - as before, $a / b^C = a / a = 1$
  - need to sum this over i…

# Proof of Master Theorem (out of scope)

> **Case 1**: f(n) << Θ($n^C$)
  - need to show summation is O($n^C$)
  - then the first term is just as big

> Time for all levels:

$$\sum_{i=0}^{\log_b n - 1} a^i f(n/b^i) \leq An^{C-\varepsilon} \sum_{i=0}^{\log_b n - 1} (b^\varepsilon)^i = An^{C-\varepsilon} \cdot n^\varepsilon = AnC$$

  - geometric series with largest term $(b^\varepsilon)^{\log_b n} = \left(b^{\log_b n}\right)^\varepsilon = \mathrm{n}^\varepsilon$

# Proof of Master Theorem (out of scope)

- **Case 3**: f(n) >> Θ($n^C$)
  - need to show summation is Θ(f(n)), which dominates first term

- Will use: a f(n / b) = C f(n) for some C < 1
- Then $a^i$ f(n / $b^i$) < $C^i$ f(n), so…

$$\sum_{i=0}^{\log_b n - 1} a^i f(n/b^i) \le f(n) \sum_{i=0}^{\log_b n - 1} C^i = f(n) \cdot O(1)$$

  - geometric series with largest term $C^0$ = 1