HW 1: Shortest Paths
CSE 417 Winter 2018

Problem 1: Designer Jeans Manufacturer[1]

The manager of a designer jeans company needs your help to determine the cheapest way to produce jeans.

Each week, he receives a list of orders for jeans of different inseam lengths, ranging from 22 to 34 inches.[2] To make the jeans, they begin by purchasing regular denim jeans from another manufacturer. Then, they repeatedly wash them, scrape them, stomp on them, etc. to give them that designer look. (See the picture on the right.)

For each size of jeans, however, the manager has a decision to make. He can either buy regular jeans of the same inseam length or he can buy jeans of a larger length and shorten (hem) them. Even though the larger regular jeans are usually more expensive and hemming is costly too, this process can be cheaper if he buys enough of the larger regular jeans because his cost per unit decreases as the size of his order increases.

Specifically, we will assume that his cost to order M_S regular jeans of size S is $A + B_S M_S$, where A is the fixed cost per order and B_S is the per-unit cost of that size of regular jeans.[3] We will also assume that the cost of hemming those jeans is H per unit. Thus, it will be cheaper to produce M_S designer jeans of size S and M_{S+1} of size S+1 by hemming the larger size if [4]

$$A + B_{S+1} (M_S + M_{S+1}) + H M_S < A + B_S M_S + A + B_{S+1} M_{S+1}.$$

Depending on the costs, A, B_S, and B_{S+1}, which change every week, one or the other can be cheaper.

The manager's problem is more complicated than this, however, because he must decide how to manufacture all the 13 different sizes of jeans — not just two, as in the example above — and he can use one order of regular jeans to produce multiple sizes of designer jeans — not just two as in the example. In total, the manager must consider the possibility of buying **any subset** of sizes of regular jeans that includes inseam size 34. That will work for him because he can get each of the intermediate sizes of regular jeans that he didn't buy by hemming the closest larger size down to that size.

We will solve this problem of finding the best subset by reducing it to a shortest path problem....

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[1] Image from http://www.oasisamor.org/torn-skinny-jeans/designer-torn-skinny-jeans-200-cute-ripped-jeans-outfits-for-winter-2017/
[2] We will assume these are women's jeans with only one size dimension, rather than men's jeans with two.
[3] Assume this formula is correct even for M_S = 0. I.e., we pay the fix cost even if we order no jeans at all.
[4] The cost to make the jeans "designer" is the same in both cases, so I have left out that part of the cost.

Our graph will have one node for each size, 34, 33, ..., 22, and a special node 21 that represents the end. It will have directed edges from each size to each *strictly smaller* size and from each size to 21.

a. Describe how to define the length of an edge (S_1, S_2) so that it properly represents the cost of ordering enough regular jeans of size S_1 to satisfy the orders for sizes S_1, $S_1 - 1$, ..., $S_2 + 1$ and hemming each of the smaller sizes ($S_1 - 1$ ... $S_2 + 1$).

b. As briefly as possible, explain why the length of the shortest path in this graph from 34 to 21 gives the cheapest way of producing jeans of all the sizes.

c. As briefly as possible, explain how to determine the cheapest way to produce the jeans from the shortest path from 34 to 21. (That is, explain how to turn the shortest path back into a prescription for how to make the jeans.)

d. After a few weeks, the manager starts getting feedback from customers that are unhappy with jeans that were produced from regular jeans 3 or more sizes larger. How can you fix the construction above (the nodes, edges, or lengths) so that it only considers producing jeans by hemming regular jeans 1–2 sizes larger. *Here, you can assume every $M_S > 0$.*

Problem 2: Designer Jeans Store

The manager of a designer jeans store needs your help to determine the cheapest way to stock the shelves with enough designer jeans to satisfy demand throughout the coming year.

For each month, she knows the number that she needs to put on shelves and the per-unit cost of buying those jeans just before the month starts. The per-unit cost of the jeans is higher in some months, like just before school starts, and lower in others, like the middle of winter.

She has a decision to make, however, about how to get the jeans she needs. She can buy them just before the month starts, as described above, but she also has the option of buying them earlier in the year (when they are cheaper) and holding them in storage until she needs them. If she holds them in storage, then she must pay an additional fee per-unit. That fee also can change each month.

For this problem, unlike the previous one, we will consider separately the question of how to get the jeans she needs for each month. For each month, we will reduce the problem of finding the cheapest per-unit cost for getting those to a shortest path problem.

For each month i (in 1 ... 12), let B_i denote the per-unit cost of buying the jeans just before the month starts. And let S_i denote the per-unit cost of storing the jeans that month.

a. We will model this as a shortest path problem with nodes for months 1—12 plus a node 0 at the start. Describe what edges you will include and what their lengths will be so that the length of any path from 0 to node k gives the per-unit cost for some way of getting jeans for month k.

b. As briefly as possible, explain why the length of the shortest path in this graph from 0 to k gives the cheapest way of getting jeans for month k.

c. As briefly as possible, explain how to determine the cheapest way to get the jeans for month k from the shortest path from 0 to k.

d. As briefly as possible, explain how to determine how much storage space we will need in month k.

e. **Bonus**: Is it possible to solve this problem for all 12 months simultaneously by finding a single shortest path like we did in Problem 1? Explain.