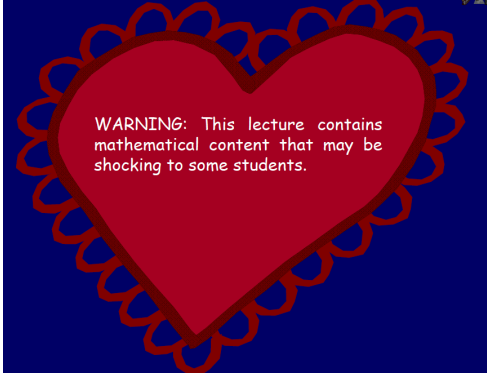


Stable Matching

Many slides due to Kevin Wayne



Picture from "Great Theoretical Ideas in Computer Science" course at CMU

15 October 2012 Nobel Prize Announcement

[The Royal Swedish Academy of Sciences has decided to award The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel for 2012 to](#)

- Alvin E. Roth
Harvard University, Cambridge, MA, USA, and Harvard Business School, Boston, MA, USA

and

- Lloyd S. Shapley
University of California, Los Angeles, CA, USA

"for the theory of stable allocations and the practice of market design".

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More detailed citation

"This year's Prize concerns a central economic problem: how to match different agents as well as possible. For example, students have to be matched with schools, and donors of human organs with patients in need of a transplant. How can such matching be accomplished as efficiently as possible? What methods are beneficial to what groups? The prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions."

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Stable Matching Problem

Goal. Given n boys and n girls, find a "suitable" matching. Participants rate members of opposite sex.

- Each boy lists girls in order of preference from best to worst.
- Each girl lists boys in order of preference from best to worst.

<p>Boys:</p> <p>X: A, B, C</p> <p>Y: B, A, C</p> <p>Z: A, B, C</p>	<p>Girls:</p> <p>A: Y, X, Z</p> <p>B: X, Y, Z</p> <p>C: X, Y, Z</p>
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- What matching makes sense?
- For example, how about X-A, Y-C, Z-B

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Stable Matching Problem

Perfect matching: everyone is matched to one person. (that will be their date.)

- Each boy is matched to exactly one girl.
- Each girl is matched to exactly one boy.

Stability: no incentive for two people not matched with each other to dump the person they are with.

- In matching, an unmatched pair $b-g^*$ is **unstable** if boy b and girl g^* prefer each other to current partners.
- Unstable pair $b-g^*$ could each improve by running off together

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n boys and n girls, find a stable matching if **one exists**.

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Understanding the Solution

For this example

<p>Boys: X: A, B, C Y: B, A, C Z: A, B, C</p>	<p>Girls: A: Y, X, Z B: X, Y, Z C: X, Y, Z</p>
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Here are two stable matchings:

X---A	X---B
Y---B	Y---A
Z---C	Z---C

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Stable Matching Problem

Q. Do stable matchings always exist?

Not clear.

How would you go about checking?

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Boy-proposing Algorithm [Gale-Shapley 1962]

Algorithm takes place over a series of days. Each day:

Morning:
 Each girl stands on her balcony. Each boy that nobody has "accepted yet" stands under the balcony of the favorite girl he has not yet ruled out and invites her to be his match.

Afternoon:
 Girls who have at least one suitor say to their favorite "Maybe, stay here and sleep under my balcony until tomorrow."
 To the other ones they say "No, I will never be your match."

Evening:
 Any boy who has been said "no" to crosses that girl off his list and goes home.

Stopping condition: As soon as all boys are either sleeping under a balcony (happy) or have crossed everyone off their list, the process stops and each girl is matched with her happy guy (if she has one).

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Run example

Some basic questions

Does the algorithm terminate or could it go on forever?

Yes, it always terminates!

Each step in which some boy gets rejected, a girl gets crossed off one of the lists.

That means that every boy ends up happy or is rejected by all the girls.

Is it possible for a boy be rejected by all the girls?

No!

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Everybody goes!

Key observation. Once a girl has a boy under her balcony, she will definitely be matched; and the sequence of boys under her balcony get better and better (according to her preference list)!

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Everybody goes!

Observation 1. Boys visit girls in decreasing order of preference.

If a boy is rejected by every girl, then he went under every single girl's balcony at some point.

Observation 2. Once a girl has a boy under her balcony, she will definitely be matched; and the sequence of boys under her balcony get better and better (according to her preference list)!

Claim. This means that each girl ends up with her absolute favorite of all the boys who come under her balcony.

Conclusion: None of the n boys is rejected by every girl and everyone ends up matched!

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The resulting matching is stable!!

Claim. No unstable pairs.

To see why not, consider any unmatched pair $b-g^*$ and argue that they cannot be an unstable pair.

Case 1: b never proposed to g^*

Case 2: b proposed to g^*

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Summary

Stable matching problem. Given n boys and n girls, and their preferences, find a stable matching if one exists.

Boy-proposing algorithm. Guarantees to find a stable matching **no matter what the preference lists are!**

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Summary

Stable matching problem. Given n boys and n girls, and their preferences, find a stable matching if one exists.

Boy-proposing algorithm. Guarantees to find a stable matching for **any** set of preference lists. And does so efficiently -number of steps is at most the sum of the lengths of the preference lists.

There can be many stable matchings. Which one does this procedure find?

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Understanding the Solution

Recall that for this example

Boys:

X: A, B, C

Y: B, A, C

Z: A, B, C

Girls:

A: Y, X, Z

B: X, Y, Z

C: X, Y, Z

There are two stable matchings:

X---A

Y---B

Z---C

X---B

Y---A

Z---C

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Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. If so, which one?

Definition: Girl g is an **achievable match** of boy b (and b is an achievable match of g) if **there exists** some stable matching in which they are matched.

Which are achievable matches of X in our previous example? Y ? Z ?

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Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. If so, which one?

Definition: Girl g is an **achievable match** of boy b if there exists some stable matching in which they are matched.

Claim: The outcome of this algorithm is boy-optimal!!
Each boy is matched with his favorite achievable girl!

The outcome of this procedure is simultaneously best for each and every boy!!!!

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Proof of male-optimality

Def. Boy b is an **achievable match** of girl g if there exists some stable matching in which they are matched.

Boy-optimal assignment. Each boy ends up with favorite achievable match!!!

Proof: Consider the first day that some boy gets rejected by his favorite achievable match, say Bob is rejected by Alice for David.

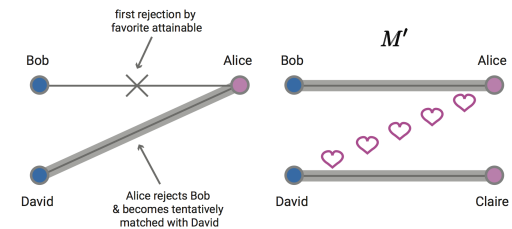
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Proof of male-optimality

Def. Boy b is an **achievable match** of girl g if there exists some stable matching in which they are matched.

Boy-optimal assignment. Each boy ends up with favorite achievable match!!!

During execution of men-proposing algorithm



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Summary so far

Stable matching problem. Given n boys and n girls, and their preferences, find a stable matching if one exists.

Boy-proposing algorithm. Guarantees to find a stable matching for **any** set of preference lists. And does so efficiently -number of steps is at most the sum of the lengths of the preference lists.

Boy-optimal assignment. Each boy ends up with favorite achievable date!!!

What about the girls? How do they fare?

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Not good....

Bad for girls!!! Each girl ends up with her **least** favorite (worst) achievable date!

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Not good....

Bad for girls!!! Each girl ends up with her **least** favorite achievable date!

M (result of men-proposing algorithm) \tilde{M}

Alice is Bob's most preferred attainable
 Dan is Alice's least preferred Attainable

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Summary

Stable matching problem. Given n boys and n girls, and their preferences, find a stable matching if one exists.

Boy-proposing algorithm. Guarantees to find a stable matching for **any** set of preference lists. And does so efficiently -number of steps is at most the sum of the lengths of the preference lists.

Boy-optimal assignment. Each boy ends up with favorite achievable date!!!

Girl-pessimal assignment. Each girl ends up with her least favorite (worst) achievable date!!!

What can the girls do about this?

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Efficient Implementation

Efficient implementation. How can we implement this algorithm in $O(n^2)$ time?

Note: this is **linear** in the size of the input.

What data structures would you use?

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Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Note: this is **linear** in the size of the input.

Representing men and women.

- Assume men are named 1, ..., n.
- Assume women are named 1', ..., n'.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
 - set entry to 0 if unmatched
 - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man m .

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Efficient Implementation

Women rejecting/accepting.

- Does woman w prefer man m to man m' ?
- For each woman, create **inverse** of preference list of men.
- Constant time access for each query after $O(n)$ preprocessing.

Amy	1 st	2 nd	3 rd	4 th	5 th	6 th	7 th	8 th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4 th	8 th	2 nd	5 th	6 th	7 th	3 rd	1 st

```
for i = 1 to n
  inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3] < inverse[6]`
2 7

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A few good people

Suppose k of the men are "good" and the rest are "bad", and similarly k of the women are "good" and the rest are "bad".

Everybody would rather marry a good person than a bad person.

Show that in every stable matching, every good man is married to a good woman.

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Honesty

Can a man or woman end up better off by lying about their preferences?

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Honesty

Can a man or woman end up better off by lying about their preferences?

	1 st	2 nd	3 rd
Xavier	C	A	B
Yancey	A	C	B
Zeus	C	A	B

	1 st	2 nd	3 rd
Amy	X	Y	Z
Bertha	X	Y	Z
Clare	Y	X	Z

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Honesty

Can a man or woman end up better off by lying about their preferences?

	1 st	2 nd	3 rd
Xavier	C	A	B
Yancey	A	C	B
Zeus	C	A	B

	1 st	2 nd	3 rd
Amy	X	Y	Z
Bertha	X	Y	Z
Clare	Y	X	Z

If Clare reports: Y, Z, X, she can improve her situation...

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Lessons Learned

Powerful ideas

- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]

- ✍ Historically, men propose to women. Why not vice versa?
- ✍ Men: propose early and often.
- ✍ Men: be more honest.
- ✍ Women: ask out the guys.

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