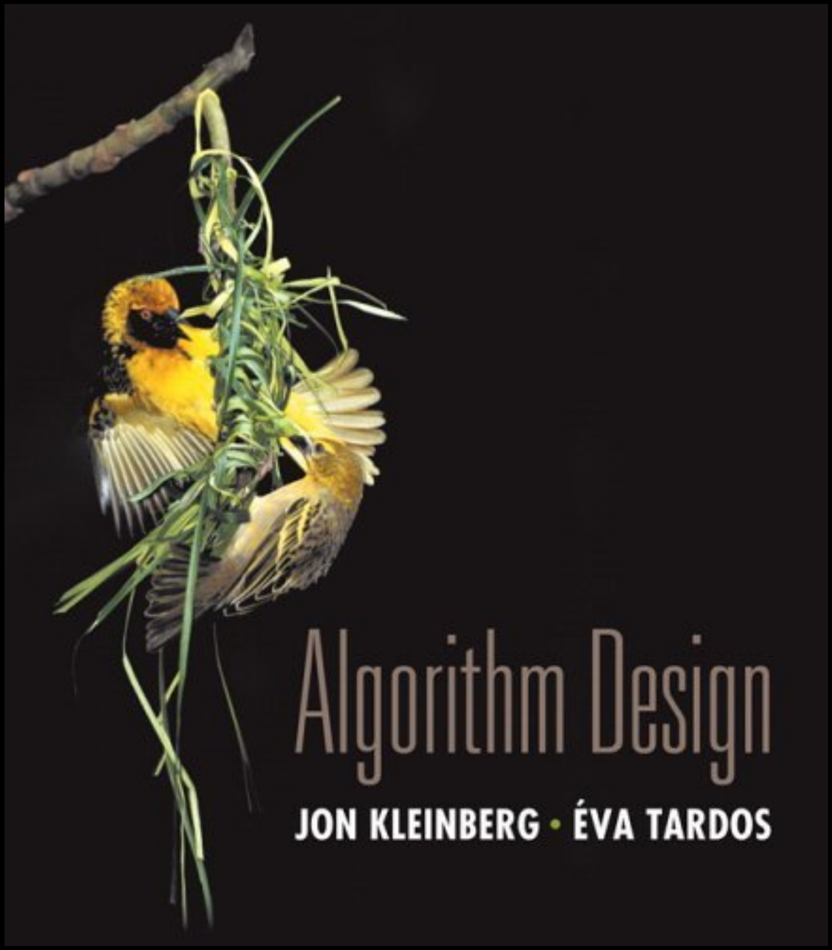

Chapter 6

Dynamic Programming

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Slides by Kevin Wayne.

Algorithmic Paradigms

Greedy. Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer. Break up a problem into sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

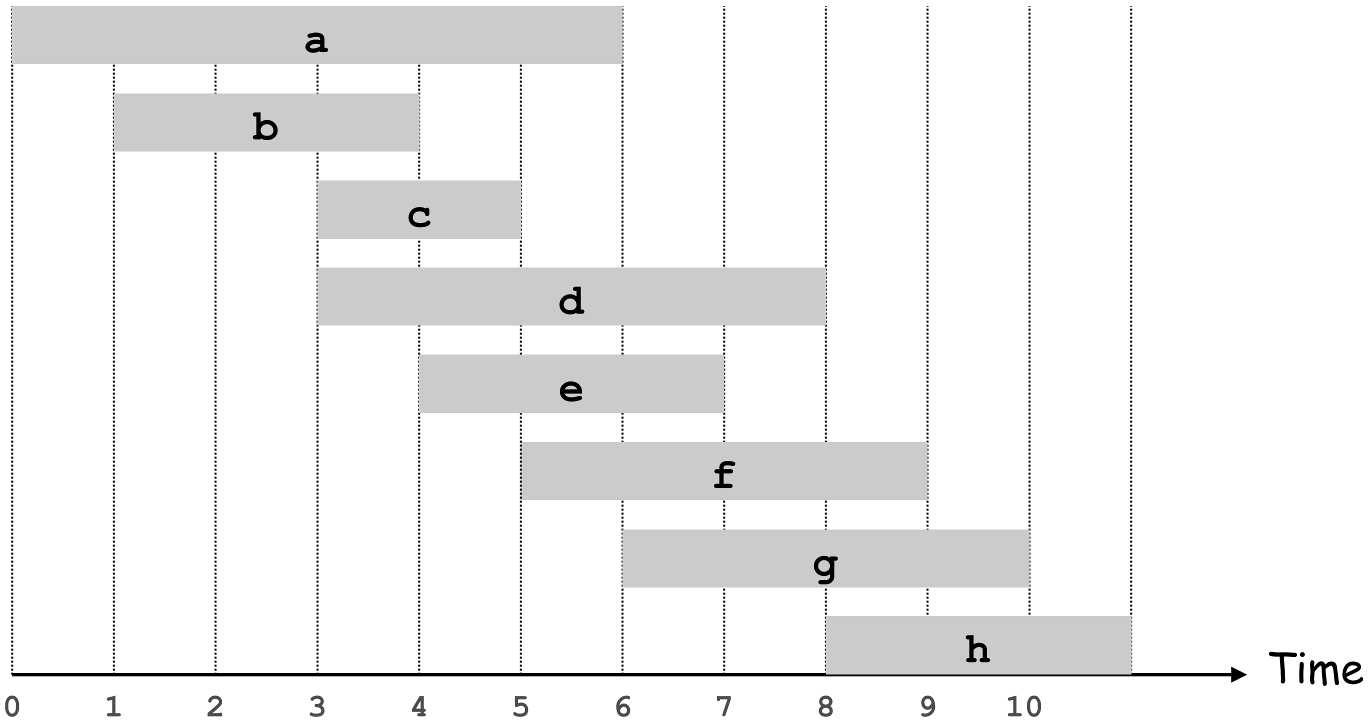
Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

6.1 Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j, finishes at f_j, and has weight or value v_j .
- Two jobs compatible if they don't overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

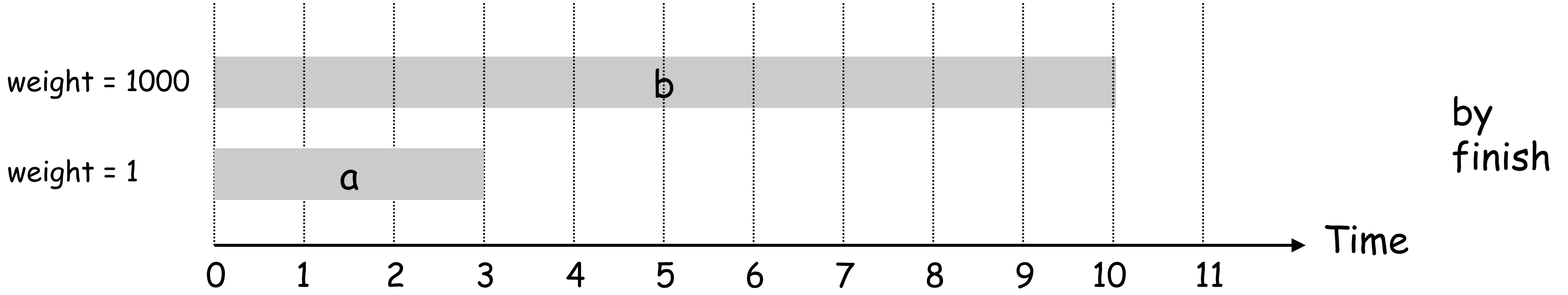


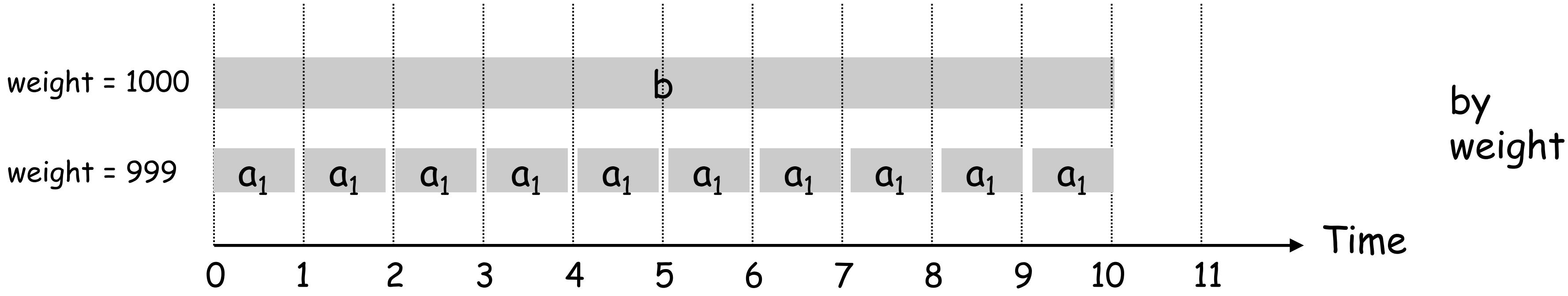
Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.



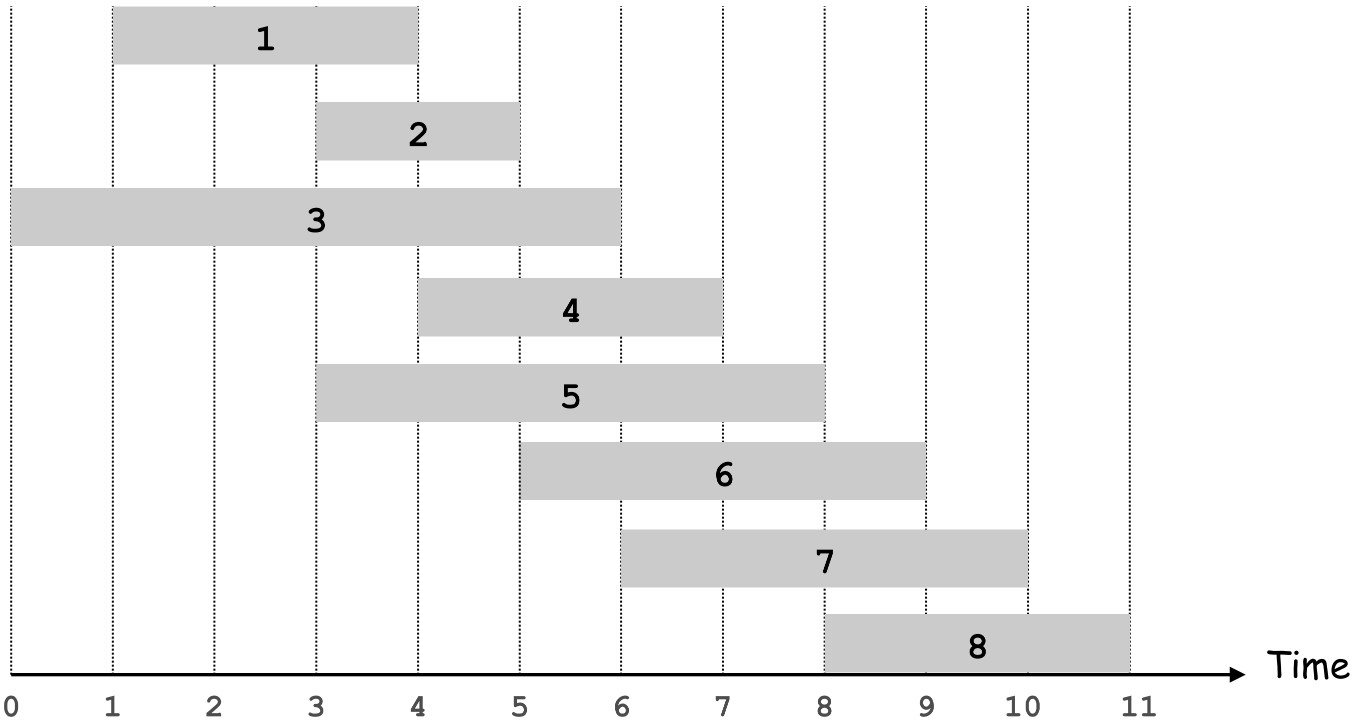


Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \le f_2 \le \ldots \le f_n$.

Def. p(j) = largest index i < j such that job i is compatible with j.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



Dynamic Programming: Binary Choice

Notation. OPT(j) = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

- Case 1: OPT selects job j.
  - collect profit v_j
  - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)

optimal substructure

- Case 2: OPT does not select job j.
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\left\{ v_j + OPT(p(j)),\ OPT(j-1) \right\} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

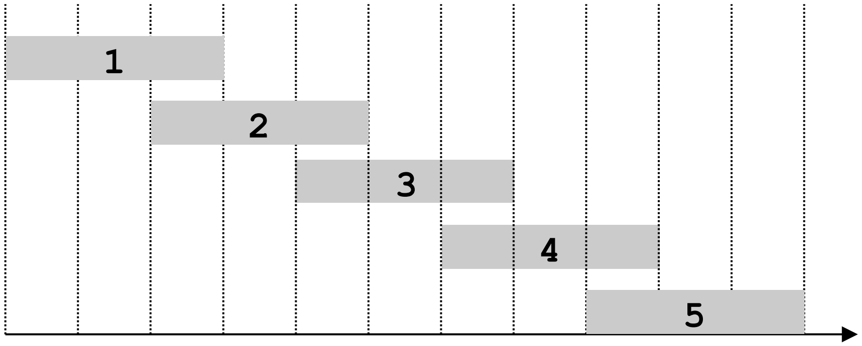
Brute force algorithm.

```
Input: n, $s_1,…,s_n$ , $f_1,…,f_n$ , $v_1,…,v_n$

Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

Compute p(1), p(2), …, p(n)

Compute-Opt(j) {
   if (j = 0)
      return 0
   else
      return max($v_j$ + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

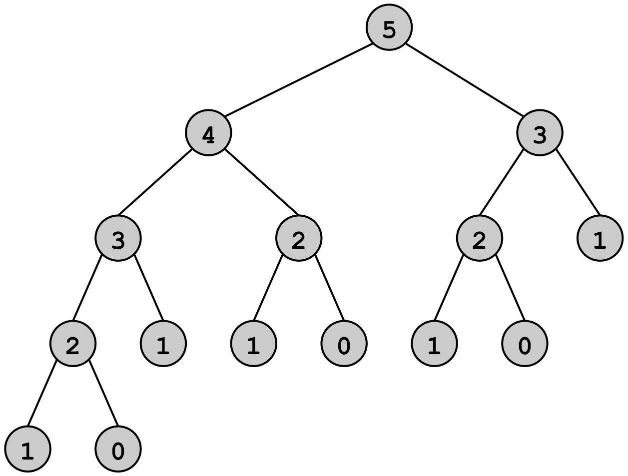
Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems ⇒ exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



p(1) = 0, p(j) = j-2



Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

```
Input: n, $s_1,…,s_n$ , $f_1,…,f_n$ , $v_1,…,v_n$

Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.
Compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty
M[0] = 0

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max($v_j$ + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

global array

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes O(n log n) time.

- Sort by finish time: O(n log n).
- Computing p(·) : O(n log n) via sorting by start time.

- `M-Compute-Opt(j)`: each invocation takes O(1) time and either
  - (i) returns an existing value `M[j]`
  - (ii) fills in one new entry `M[j]` and makes two recursive calls

- Progress measure Φ = # nonempty entries of `M[]`.
  - initially $\Phi = 0$, throughout $\Phi \leq n$.
  - (ii) increases Φ by 1 $\Rightarrow$ at most 2n recursive calls.

- Overall running time of `M-Compute-Opt(n)` is O(n). ▪

Remark. O(n) if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes O(n log n) time.

- Sort by finish time: O(n log n).
- Computing p(·) : O(n) after sorting by start time.
- `M-Compute-Opt(j)`: each invocation takes O(1) time and either
  - (i) returns an existing value `M[j]`
  - (ii) fills in one new entry `M[j]` and makes two recursive calls
- Progress measure Φ = # nonempty entries of `M[]`.
  - initially $\Phi = 0$, throughout $\Phi \leq n$.
  - (ii) increases Φ by 1 $\Rightarrow$ at most 2n recursive calls.
- Overall running time of `M-Compute-Opt(n)` is O(n). ▪

(A bit subtle – skipping details)

Remark. O(n) if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

```
Input: n, $s_1,…,s_n$ , $f_1,…,f_n$ , $v_1,…,v_n$

Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max($v_j$ + M[p(j)], M[j-1])
}

Output M[n]
```

Claim: M[j] is value of optimal solution for jobs 1..j

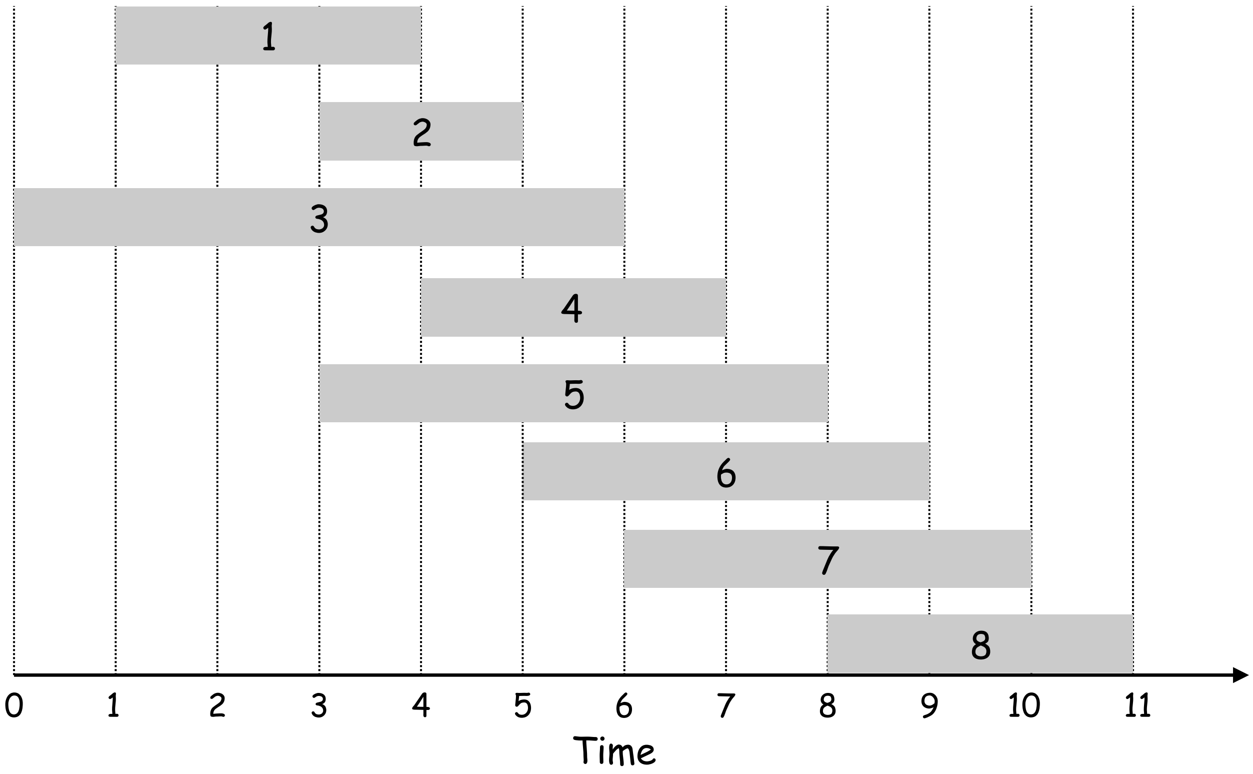
Timing: Easy. Main loop is O(n); sorting is O(n log n)

Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq . . . \leq f_n$.

Def. p(j) = largest index i < j such that job i is compatible with j.

Ex: p(8) = 5, p(7) = 3, p(2) = 0.



j	vj	pj	optj
0	-	-	0
1	5		
2	4		
3	2		
4	4		
5	3		
6	2		
7	8		
8	4		

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing – "traceback"

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if ($v_j$ + M[p(j)] > M[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

the condition determining the max when computing M[ ]

the relevant sub-problem

- # of recursive calls ≤ n ⇒ O(n).

Dynamic Programming – iterative approach

Have a collection of subproblems that satisfy a few basic properties:

- Only polynomially many.
- The solution to the original problem can be easily computed from the solutions to the subproblems.
- There is a natural ordering on subproblems from "smallest" to "largest" together with an easy to compute recurrence that allows us to determine the solution to a subproblme from the solution to some number of smaller subproblems.

6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

`W = 11`

#	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i.

Ex: { 5, 2, 1 } achieves only value = 35 ⇒ greedy not optimal.

Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 }
- Case 2: OPT selects item i.
  - accepting item i does not immediately imply that we will have to reject other items
  - without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 } using weight limit w

- Case 2: OPT selects item i.
  - new weight limit = w – w_i
  - OPT selects best of { 1, 2, …, i–1 } using this new weight limit

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ \ v_i + OPT(i-1,w-w_i)\,\} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n-by-W array.
M(i, w) = max profit subset of items 1, …, i with weight limit w.

```
Input: n, W, $w_1$,…,$w_N$, $v_1$,…,$v_N$

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if ($w_i$ > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], $v_i$ + M[i-1, w-$w_i$ ]}

return M[n, W]
```

M(i, w) = max profit subset of items 1, …, i with weight limit w.

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

```
Input: n, W, $w_1,…,w_N$, $v_1,…,v_N$

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if ($w_i$ > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], $v_i$ + M[i-1, w-$w_i$ ]}

return M[n, W]
```

W = 11

Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

M(i, w) = max profit subset of items 1, …, i with weight limit w.

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

W = 11

```
Input: n, W, $w_1$,…,$w_N$, $v_1$,…,$v_N$

for w = 0 to W
   M[0, w] = 0

for i = 1 to n
   for w = 1 to W
      if ($w_i$ > w)
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max {M[i-1, w], $v_i$ + M[i-1, w-$w_i$ ]}

return M[n, W]
```

How do you find the actual solution once you're filled out the table?

Dynamic Programming – iterative approach

Have a collection of subproblems that satisfy a few basic properties:

- Only polynomially many (hopefully)
- The solution to the original problem can be easily computed from the solutions to the subproblems.
- There is a natural ordering on subproblems from "smallest" to "largest" together with an easy to compute recurrence that allows us to determine the solution to a subproblme from the solution to some number of smaller subproblems.

Knapsack Problem: Running Time

Running time. $\Theta(n\ W)$.

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete. [Chapter 8]

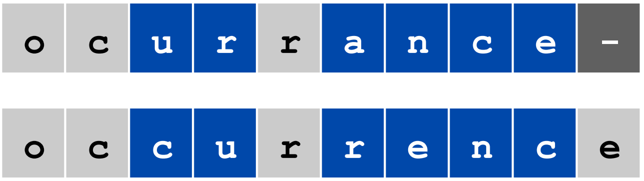
Knapsack approximation algorithm. There exists a poly-time algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

6.6 Sequence Alignment

String Similarity

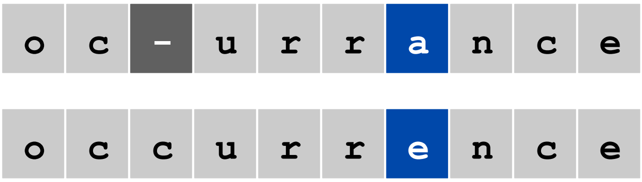
How similar are two strings?

- **`ocurrance`**
- **`occurrence`**



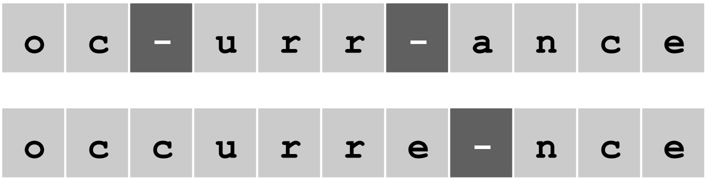
o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap



o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap



o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e

0 mismatches, 3 gaps

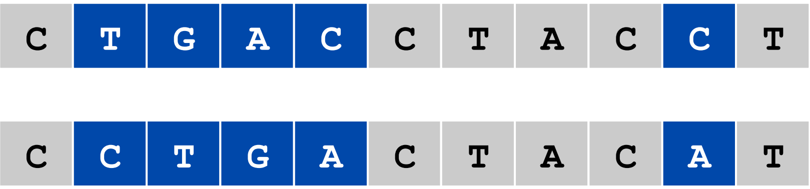
Edit Distance

Applications.

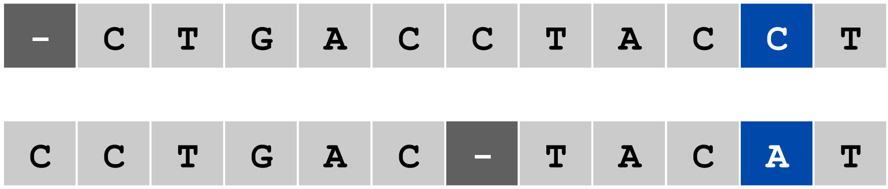
- Basis for Unix diff.
- Speech recognition.
- Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ; mismatch penalty α_{pq}.
- Cost = sum of gap and mismatch penalties.



$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$



$2\delta + \alpha_{CA}$

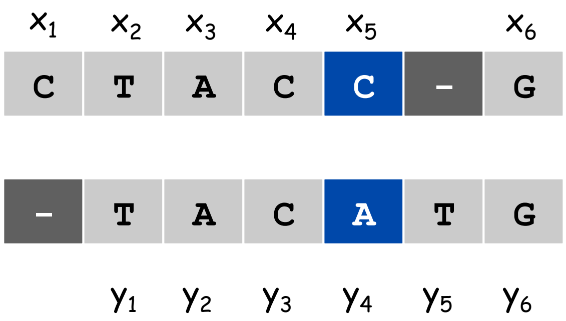
Sequence Alignment

Goal: Given two strings $X = x_1\, x_2\, .\, .\, .\, x_m$ and $Y = y_1\, y_2\, .\, .\, .\, y_n$ find alignment of minimum cost.

Def. An alignment M is a set of ordered pairs x_i-y_j such that each item occurs in at most one pair and no crossings.

Def. The pair x_i-y_j and $x_{i'}$-$y_{j'}$ cross if i < i', but j > j'. Don't allow crossing.

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i\,:\,x_i \text{ unmatched}} \delta + \sum_{j\,:\,y_j \text{ unmatched}} \delta}_{\text{gap}}$$



Ex: `CTACCG` vs. `TACATG`.

Sol: $M = x_2\text{-}y_1, x_3\text{-}y_2, x_4\text{-}y_3, x_5\text{-}y_4, x_6\text{-}y_6$.

Sequence Alignment: Problem Structure

Def. OPT(i, j) = min cost of aligning strings $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_j$.

- Case 1: OPT matches x_i-y_j.
  - pay mismatch for x_i-y_j + min cost of aligning two strings $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_{j-1}$
- Case 2a: OPT leaves x_i unmatched.
  - pay gap for x_i and min cost of aligning $x_1 x_2 \ldots x_{i-1}$ and $y_1 y_2 \ldots y_j$
- Case 2b: OPT leaves y_j unmatched.
  - pay gap for y_j and min cost of aligning $x_1 x_2 \ldots x_i$ and $y_1 y_2 \ldots y_{j-1}$

$$OPT(i, j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence Alignment: Algorithm

```
Sequence-Alignment(m, n, $x_1x_2...x_m$, $y_1y_2...y_n$, $\delta$, $\alpha$) {
   for i = 0 to m
      M[i, 0] = i$\delta$
   for j = 0 to n
      M[0, j] = j$\delta$

   for i = 1 to m
      for j = 1 to n
         M[i, j] = min($\alpha$[$x_i$, $y_j$] + M[i-1, j-1],
                       $\delta$ + M[i-1, j],
                       $\delta$ + M[i, j-1])
   return M[m, n]
}
```

Analysis. $\Theta(mn)$ time and space.

English words or sentences: m, n $\le$ 10.

Computational biology: m = n = 100,000. 10 billions ops OK, but 10GB array?

Dynamic Programming – iterative/bottom-up approach

Have a collection of subproblems that satisfy a few basic properties:

- Only polynomially many.
- The solution to the original problem can be easily computed from the solutions to the subproblems.
- There is a natural ordering on subproblems from “smallest” to “largest” together with an easy to compute recurrence that allows us to determine the solution to a subproblme from the solution to some number of smaller subproblems.