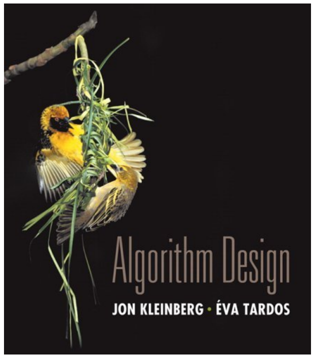


Introduction:
Some Representative Problems

Slides have been slightly modified by Anna Karlin

1

Stable Matching

15 October 2012 Nobel Prize Announcement

The Royal Swedish Academy of Sciences has decided to award The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel for 2012 to

- **Alvin E. Roth**
  Harvard University, Cambridge, MA, USA, and Harvard Business School, Boston, MA, USA

and

- **Lloyd S. Shapley**
  University of California, Los Angeles, CA, USA

"for the theory of stable allocations and the practice of market design".

3

More detailed citation

"This year's Prize concerns a central economic problem: how to match different agents as well as possible. For example, students have to be matched with schools, and donors of human organs with patients in need of a transplant. How can such matching be accomplished as efficiently as possible? What methods are beneficial to what groups? The prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions."

4

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

- What matching makes sense?

5

Stable Matching Problem

Q. Does matching X-C, Y-B, Z-A make sense?

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

6

Stable Matching Problem

Q Does matching X-C, Y-B, Z-A make sense?
A. No. Bertha and Xavier will hook up.

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

7

Stable Matching Problem

Q. Does matching X-A, Y-B, Z-C make sense?

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ … least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

8

Stable Matching Problem

Perfect matching: everyone is matched monogamously.
- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for two unmatched people to run off with each other.
- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Roommate Problem

Q. Do stable matchings always exist?
A. Not clear.

Stable roommate problem.
- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

Stable Roommate Problem

Q. Do stable matchings always exist?
A. Not clear.

Stable roommate problem.
- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
Adam	B	C	D
Bob	C	A	D
Chris	A	B	D
Doofus	A	B	C

A-B, C-D $\Rightarrow$ B-C unstable
A-C, B-D $\Rightarrow$ A-B unstable
A-D, B-C $\Rightarrow$ A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem. But they do exist for stable marriage problem!

Men-proposing Algorithm [Gale-Shapley 1962]

```
Algorithm takes place over a series of days. Each day:

Morning:
Each girl stands on her balcony. One (arbitrary) boy that
nobody has "accepted yet" stands under the balcony of the
favorite girl he has not yet ruled out and proposes to her.

Afternoon:
Girls who have at least one suitor say to their favorite
"Maybe, stay here and sleep under my balcony until tomorrow."
To the other one they say "No, I will never marry you."

Evening:
Any boy who has been said "no" to crosses that girl off his
list and goes home.

Termination condition: If all boys are sleeping under a
balcony, the process stops and each girl marries her suitor.
```

Proof of Correctness: Termination

Why?

Claim. Algorithm terminates after at most n^2 proposals.
Pf. Each step in which some man gets rejected, a woman gets crossed off one of the lists. The total length of all the lists is only n^2 . ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

$n(n-1) + 1$ proposals required

13

Proof of Correctness: Perfection

Claim. All men and women get matched.

Why?

14

Some crucial observations

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. All men and women get matched.

15

Proof of Correctness: Perfection

Claim. All men and women get matched.
Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

16

It works!!!

Claim. No unstable pairs.

Proof of Correctness: Stability

Claim. No unstable pairs.
Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in matching S* output by men-proposing algorithm.

- Case 1: Z never proposed to A. ← men propose in decreasing order of preference
  ⇒ Z prefers his GS partner to A.
  ⇒ A-Z is stable.

- Case 2: Z proposed to A.
  ⇒ A rejected Z (right away or later)
  ⇒ A prefers her GS partner to Z. ← women only trade up
  ⇒ A-Z is stable.

- In either case A-Z is stable, a contradiction. ▪

S*
Amy-Yancey
Bertha-Zeus
. . .

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Men-proposing algorithm. Guarantees to find a stable matching for any problem instance. And does so efficiently –approximately n^2 time in the worst case.

Efficiency of algorithms digression…

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Men-proposing algorithm. Guarantees to find a stable matching for any problem instance. And does so efficiently –approximately n^2 time in the worst case.

Q. If there are multiple stable matchings, which one does this algorithm find?

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Which one does this algorithm find?

An instance with two stable matchings.
- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. If so, which one?

Def. Man m is an attainable partner of woman w if there exists some stable matching in which they are matched.

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. If so, which one?

Def. Man m is an attainable partner of woman w if there exists some stable matching in which they are matched.

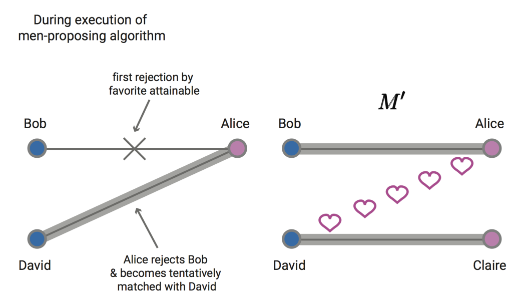
Man-optimal assignment. Each man receives best attainable partner!!!

Claim. All executions of algorithm yield man-optimal assignment, which is a stable matching!
- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best **for each and every man**.

Proof of male-optimality

Def. Man m is an attainable partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives favorite attainable partner!!!



Stable Matching Summary So Far

Stable matching problem. Given preferences of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Men-proposing algorithm. Finds a stable matching efficiently – approximately n^2 time in the worst case.

Man-optimality. In men-proposing algorithm, each man receives favorite attainable partner.

w is an attainable partner of m if there exist some stable matching where m and w are paired

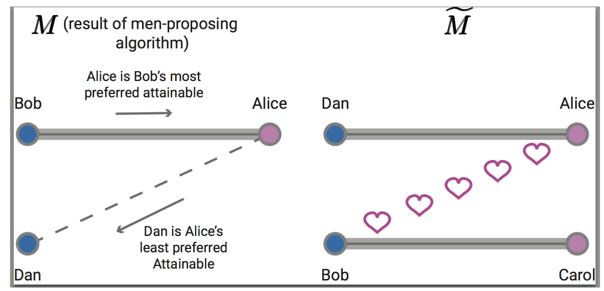
Q. What about the women?

25

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst attainable partner.

Claim. GS finds woman-pessimal stable matching S*.



26

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.
Note: this is linear in the size of the input.

Representing men and women.
- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.
- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched
  - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.
- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man `m`.

27

Efficient Implementation

Women rejecting/accepting.
- Does woman `w` prefer man `m` to man `m'`?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3]` < `inverse[6]`
2 7

28

Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Def. Matching S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

29

A bit of history...

30

History

1900

- Idea of hospitals having residents (then called "interns")

Over the next few decades

- Intense competition among hospitals for an inadequate supply of residents
  - Each hospital makes offers independently
  - Process degenerates into a race. Hospitals are steadily advancing date at which they finalize binding contracts

31

History

1944 Absurd Situation. Appointments being made 2 years ahead of time!

- All parties were unhappy
- Medical schools stop releasing any information about students before some reasonable date
- Did this fix the situation?

32

History

1944 Absurd Situation. Appointments being made 2 years ahead of time!

- All parties were unhappy
- Medical schools stop releasing any information about students before some reasonable date
- **Offers were made at a more reasonable date, but new problems developed**

History

1945-1949 Just As Competitive

- Hospitals started putting time limits on offers
- Time limit gets down to 12 hours
- Lots of unhappy people
- Many instabilities resulting from lack of cooperation

History

1950 Centralized System

- Each hospital ranks residents
- Each resident ranks hospitals
- National Resident Matching Program produces a pairing

Whoops! The pairings were not always stable. By 1952 the algorithm was GS (hospital-optimal) and therefore stable.

History Repeats Itself!
NY TIMES, March 17, 1989

The once decorous process by which federal judges select their law clerks has degenerated into a free-for-all in which some of the judges scramble for the top law school students . . .

The judge have steadily pushed up the hiring process . . .

Offered some jobs as early as February of the second year of law school . . .

On the basis of fewer grades and flimsier evidence . . .

NY TIMES

"Law of the jungle reigns . . "

The association of American Law Schools agreed not to hire before September of the third year . . .

Some extend offers from only a few hours, a practice known in the clerkship vernacular as a "short fuse" or a "hold up".

Judge Winter offered a Yale student a clerkship at 11:35 and gave her until noon to accept . . . At 11:55 . . he withdrew his offer

37

A few good people

Suppose k of the men are "good" and the rest are "bad", and similarly k of the women are "good" and the rest are "bad".

Everybody would rather marry a good person than a bad person.

Show that in every stable matching, every good man is married to a good woman.

38

Honesty

Can a man or woman end up better off by lying about their preferences?

39

Honesty

Can a man or woman end up better off by lying about their preferences?

	1st	2nd	3rd
Xavier	C	A	B
Yancey	A	C	B
Zeus	C	A	B

	1st	2nd	3rd
Amy	X	Y	Z
Bertha	X	Y	Z
Clare	Y	X	Z

40

Honesty

Can a man or woman end up better off by lying about their preferences?

	1st	2nd	3rd
Xavier	C	A	B
Yancey	A	C	B
Zeus	C	A	B

	1st	2nd	3rd
Amy	X	Y	Z
Bertha	X	Y	Z
Clare	Y	X	Z

If Clare reports: Y, Z, X,   she can improve her situation...

41

Lessons Learned

Powerful ideas
- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]
- Historically, men propose to women. Why not vice versa?
- Men: propose early and often.
- Men: be more honest.
- Women: ask out the guys.

42

Stable Matching with some forbidden pairs

Forbidden pairs not allowed to get married.
Each woman ranks all men she is not forbidden from marrying.
Each man ranks all women he is not forbidden from marrying.

A matching is stable if
- none of the usual instabilities
- no forbidden matches
- if m ends up unmatched, (m',w') are matched, and (m, w') isn't forbidden, then w' prefers m' to m. (no single man more desirable and not forbidden)
- (no single woman more desirable and not forbidden).
- if man m and woman w are both single, then (m,w) is forbidden.

- Show how to adapt the algorithm to find a stable matching in this setting.

43