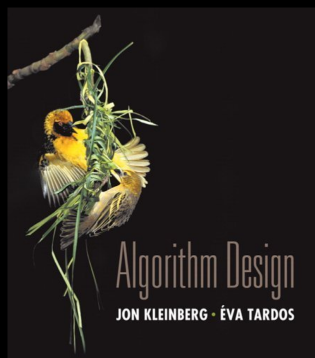


Chapter 4

Greedy Algorithms

Intro: Coin Changing

Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

Ex: 34¢.

Algorithm is "Greedy": One large coin better than two or more smaller ones

Cashier's algorithm. At each iteration, give the *largest* coin valued ≤ the amount to be paid.

Ex: $2.89.

3

Coin-Changing: Does Greedy Always Work?

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.
- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.

Algorithm is "Greedy", but also short-sighted – attractive choice now may lead to dead ends later.

Correctness is key!

4

Outline & Goals

"Greedy Algorithms"
- what they are

Pros
- intuitive
- often simple
- often fast

Cons
- often incorrect!

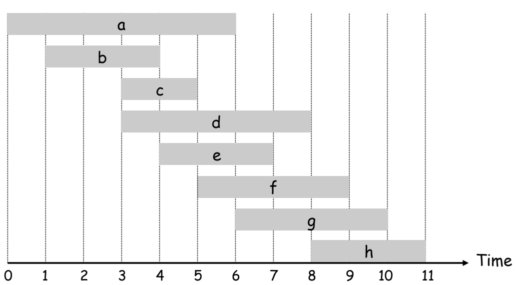
Proof techniques
- stay ahead
- structural
- exchange arguments

4.1 Interval Scheduling

Proof Technique 1: "greedy stays ahead"

Interval Scheduling

Interval scheduling.
- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- What order?
- Does that give best answer?
- Why or why not?
- Does it help to be greedy about order?

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

[Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.

[Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

[Earliest start time] Consider jobs in ascending order of start time s_j.

[Earliest finish time] Consider jobs in ascending order of finish time f_j.

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Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

    jobs selected
A ← φ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

Implementation. O(n log n).
- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \geq f_{j^*}$.

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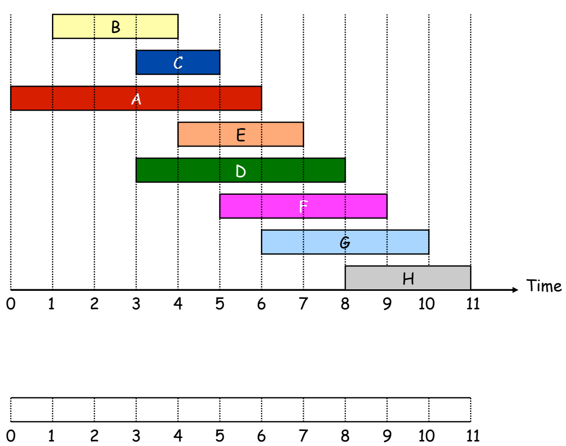
Interval Scheduling



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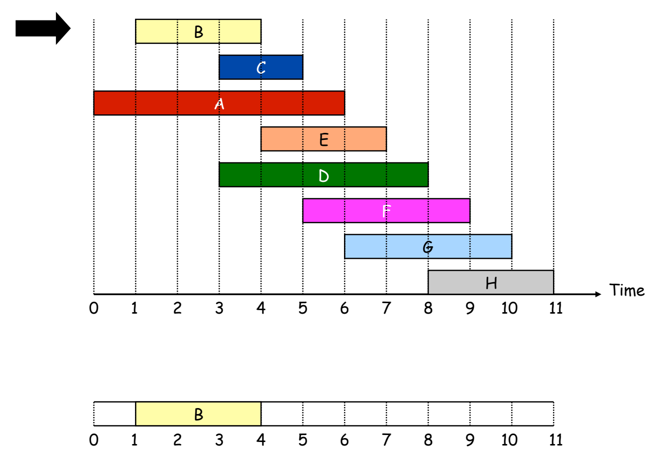
Interval Scheduling



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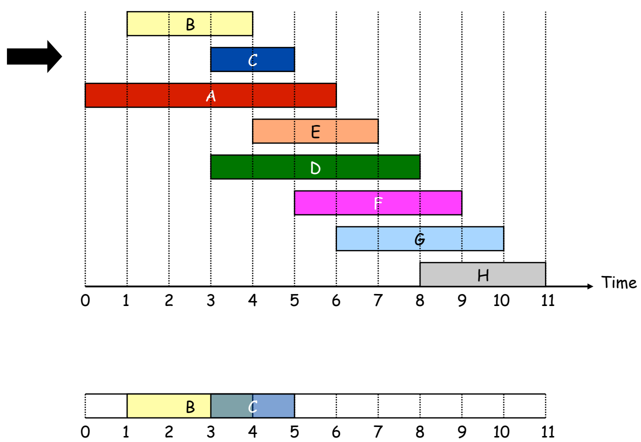
Interval Scheduling



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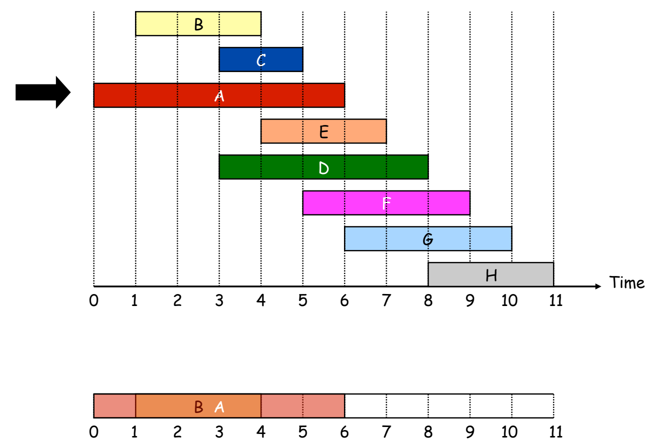
Interval Scheduling



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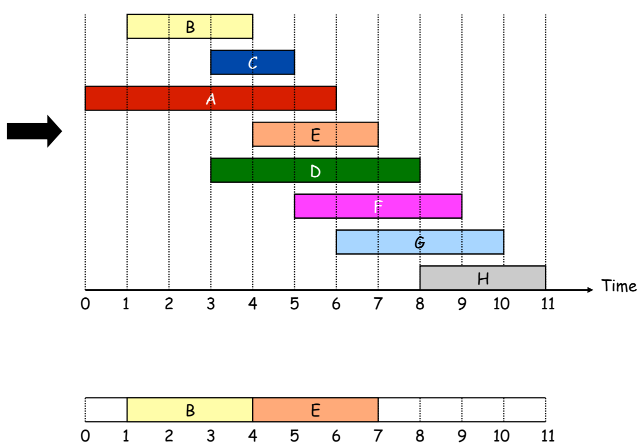
Interval Scheduling



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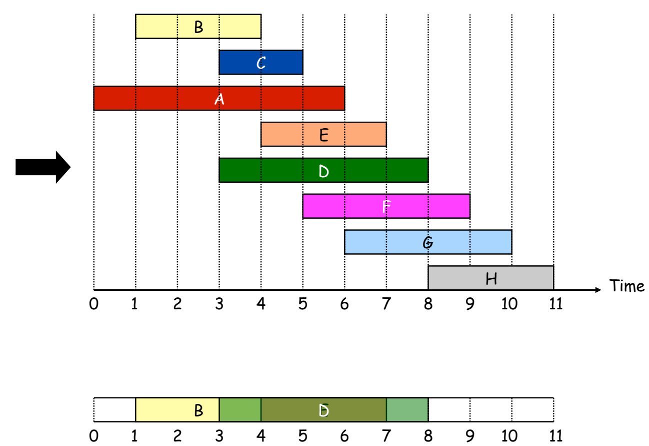
Interval Scheduling



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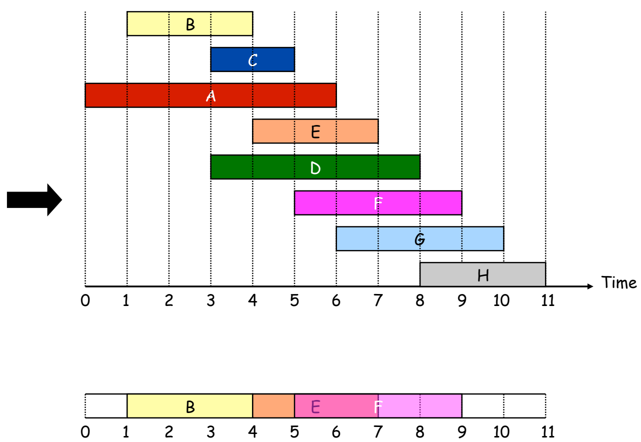
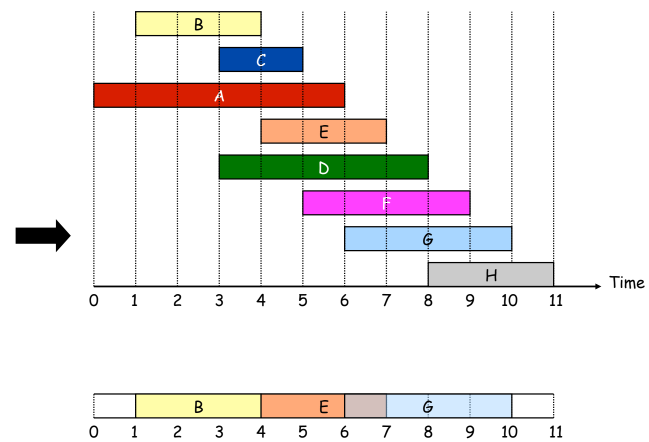
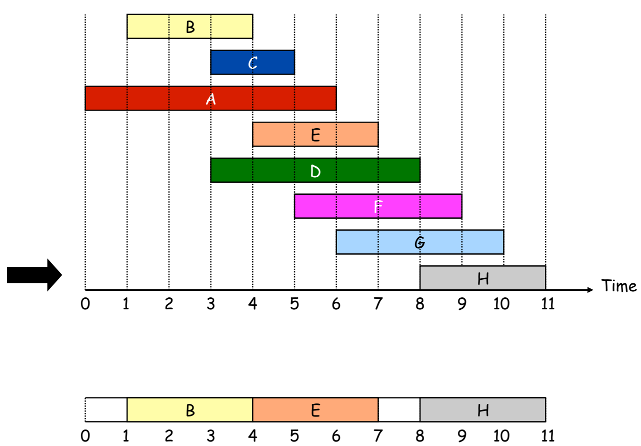
Interval Scheduling



Interval Scheduling



Interval Scheduling



Interval Scheduling: Correctness

Theorem. Greedy algorithm is optimal.

Pf. ("greedy stays ahead")

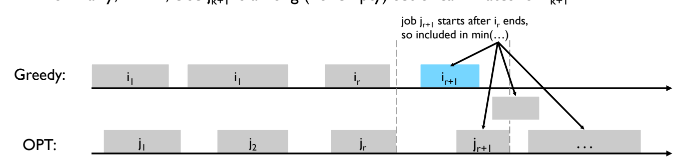
Let $i_1, i_2, \ldots i_k$ be jobs picked by greedy, $j_1, j_2, \ldots j_m$ those in some optimal solution

Show $f(i_r) \le f(j_r)$ by induction on r.

Basis: i_1 chosen to have min finish time, so $f(i_1) \le f(j_1)$

Ind: $f(i_r) \le f(j_r) \le s(j_{r+1})$, so j_{r+1} is among the candidates considered by greedy when it picked i_{r+1}, & it picks min finish, so $f(i_{r+1}) \le f(j_{r+1})$

Similarly, $k \ge m$, else j_{k+1} is among (nonempty) set of candidates for i_{k+1}



4.2 Scheduling to Minimize Lateness

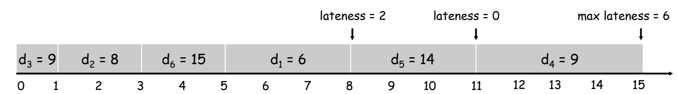
Proof Technique 2: "Exchange" Arguments

Scheduling to Minimize Lateness

Minimizing lateness problem.
- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first]
Consider jobs in ascending order of processing time t_j.

[Smallest slack]
Consider jobs in ascending order of slack $d_j - t_j$.

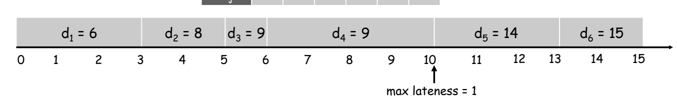
[Earliest deadline first]
Consider jobs in ascending order of deadline d_j.

Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

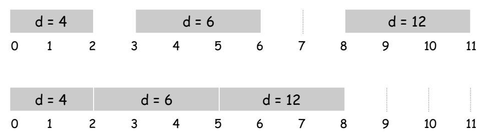
```
Sort n jobs by deadline so that d1 ≤ d2 ≤ … ≤ dn

t ← 0
for j = 1 to n
   // Assign job j to interval [t, t + tj]:
   sj ← t, fj ← t + tj
   t ← t + tj
output intervals [sj, fj]
```

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: No Idle Time

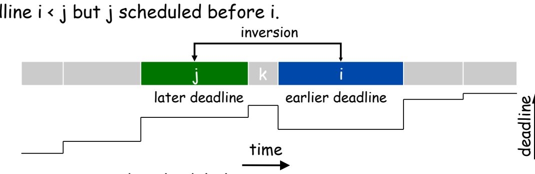
Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. An *inversion* in schedule S is a pair of jobs i and j such that: deadline i < j but j scheduled before i.



Observation. Greedy schedule has no inversions.

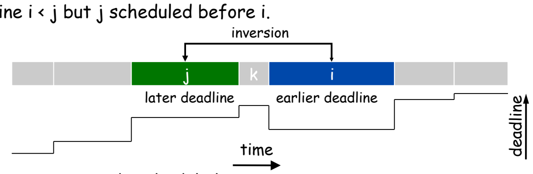
Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.

Observation. Swapping *adjacent* inversion reduces # inversions by 1

(exactly)

Minimizing Lateness: Inversions

Def. An *inversion* in schedule S is a pair of jobs i and j such that: deadline i < j but j scheduled before i.



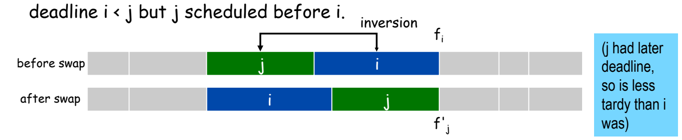
Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.
(If j & i aren't consecutive, then look at the job k scheduled right after j. If $d_k < d_j$, then (j,k) is a consecutive inversion; if not, then (k,i) is an inversion, & nearer to each other - repeat.)

Observation. Swapping *adjacent* inversion reduces # inversions by 1 (exactly)

Minimizing Lateness: Inversions

Def. An *inversion* in schedule S is a pair of jobs i and j such that: deadline i < j but j scheduled before i.

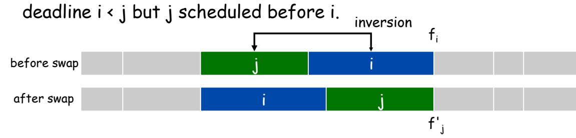


Claim. *Swapping* two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf.

Minimizing Lateness: Inversions

Def. An *inversion* in schedule S is a pair of jobs i and j such that: deadline i < j but j scheduled before i.



(j had later deadline, so is less tardy than i was)

Claim. *Swapping* two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.
- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$
- If job j is now late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (d_i \leq d_j) \\ &= \ell_i && \text{(definition)} \end{aligned}$$

only j moves later, but it's no later than i was, so max not increased

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Minimizing Lateness: Correctness of Greedy Algorithm

Theorem. Greedy schedule S is optimal

Pf. Let S* be an optimal schedule with the fewest number of inversions
Can assume S* has no idle time.
If S* has an inversion, let i-j be an adjacent inversion
Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
This contradicts definition of S*
So, S* has no inversions. But then Lateness(S) = Lateness(S*)

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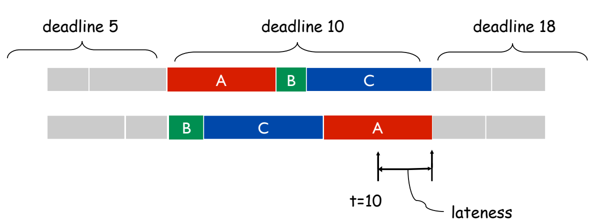
Minimizing Lateness: No Inversions

Claim. All inversion-free schedules S have the same max lateness

Pf. If S has no inversions, then deadlines of scheduled jobs are monotonically nondecreasing, i.e., they increase (or stay the same) as we walk through the schedule from left to right.
Two such schedules can differ only in the order of jobs with the same deadlines.
Within a group of jobs with the same deadline, the max lateness is the lateness of the last job in the group - order within the group doesn't matter.



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Greedy Analysis Strategies

- Solve some special cases.
- Guess at some algorithms that might work.
- Try to distinguish between them by coming up with inputs on which they do different things.

Once you have a plausible candidate, try one of the following strategies for proving optimality:

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as "good" as any other algorithm's. (Part of the cleverness is deciding what's "good.")

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality

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Problem

Given sequence S of n purchases at a stock exchange, possibly containing some events multiple times.
e.g.
Buy Amazon, Buy Google, Buy eBay, Buy Google, Buy Google, Buy Oracle

And another sequence S' of m purchases: Determine if S' is a subsequence of S in linear time.

Problem

You have n jobs $J_1, J_2, \ldots J_n$, each consisting of two stages:
- Preprocessing stage on a supercomputer
- Finishing stage on a PC

Second stage can be done in parallel (first stage has to be done sequentially.

- Job J_i needs p_i seconds of time on the supercomputer followed by f_i seconds of time on a PC.

Design an algorithm that finds a schedule (order in which to process on supercomputer) that minimizes the completion time of the last job.