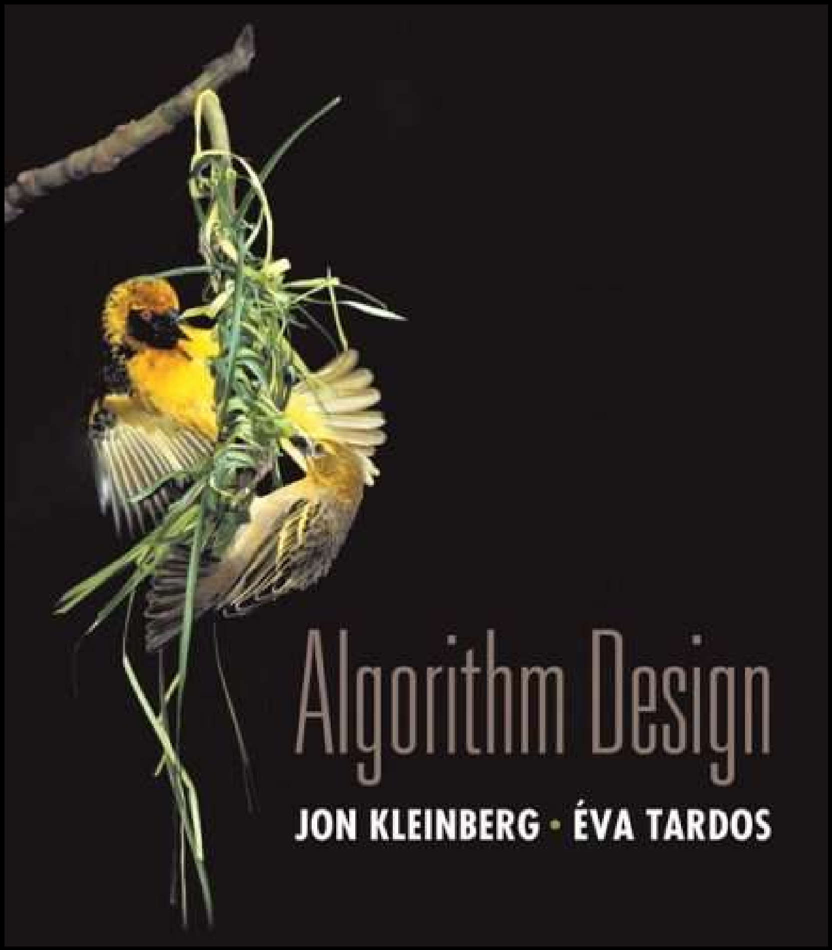


Divide and Conquer

Reading: 5.1, 5.4-5.5, 13.5

Some of the slides were
Adapted from Paul Beame

Divide-and-Conquer

Divide-and-conquer.

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Most common usage.

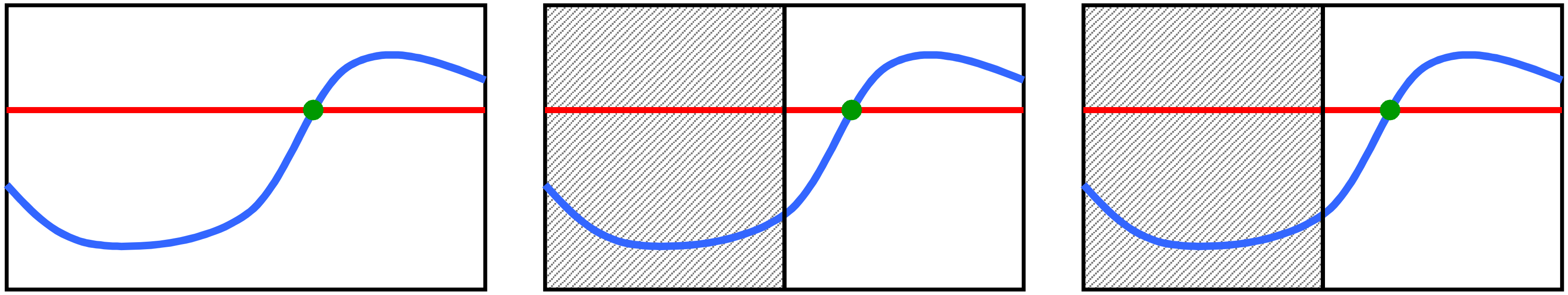
- Break up problem of size n into two equal parts of size $\frac{1}{2}n$.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: n^2.
- Divide-and-conquer: n log n.

> Divide et impera.
> Veni, vidi, vici.
> \- *Julius Caesar*

Binary search for roots (bisection method)



Given:

- continuous function f and two points $a<b$ with $f(a) \leq 0$ and $f(b) > 0$

Find:

- approximation to c s.t. $f(c)=0$ and $a \leq c < b$

Bisection method

```
Bisection(a, b, ε)
 if (a-b) < ε  then
          return(a)
 else
      c ←(a+b)/2
      if  f(c) ≤ 0 then
          return(Bisection(c, b, ε))
      else
          return(Bisection(a, c, ε))
```

Time Analysis:

At each step we halved the size of the interval

It started at size b-a

It ended at size ε

\# of calls to f is $\log_2((b-a)/\varepsilon)$

Old favorites

Binary search

- One subproblem of half size plus one comparison
- Recurrence $T(n) = T(\lceil n/2 \rceil)+1$ for $n \geq 2$
  $T(1) = 0$

  So $T(n)$ is $\lceil \log_2 n \rceil + 1$

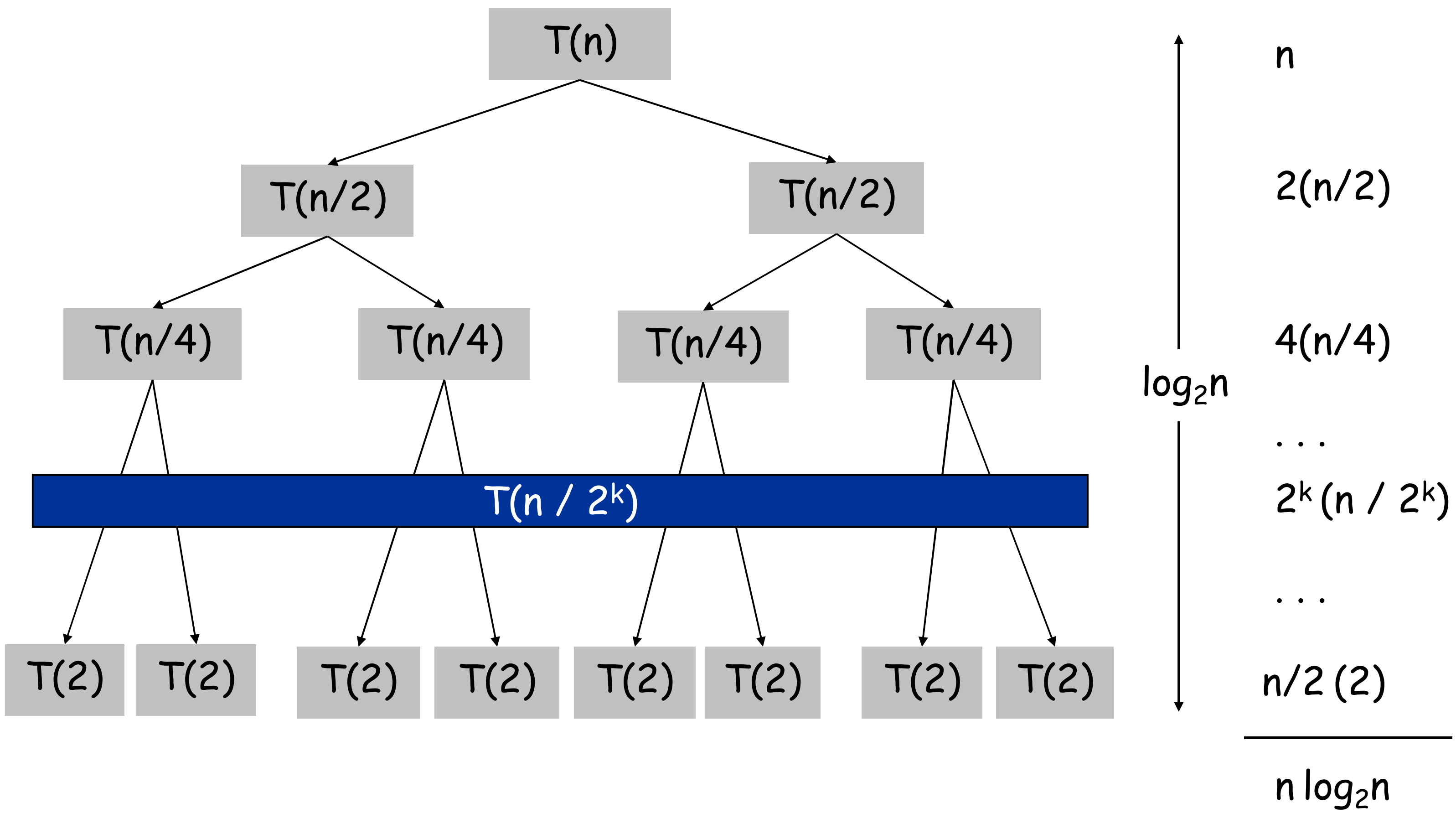
Mergesort

- Two subproblems of half size plus merge cost of n-1 comparisons
- Recurrence $T(n) \leq 2T(\lceil n/2 \rceil)+n-1$ for $n \geq 2$
  $T(1) = 0$

  Roughly n comparisons at each of $\log_2 n$ levels of recursion
  So $T(n)$ is roughly $2n \log_2 n$

Proof by Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Proof by Telescoping

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. For n > 1:

$$\begin{aligned}
\frac{T(n)}{n} &= \frac{2T(n/2)}{n} + 1 \\
&= \frac{T(n/2)}{n/2} + 1 \\
&= \frac{T(n/4)}{n/4} + 1 + 1 \\
&\cdots \\
&= \frac{T(n/n)}{n/n} + \underbrace{1 + \cdots + 1}_{\log_2 n} \\
&= \log_2 n
\end{aligned}$$

Proof by Induction

Claim. If T(n) satisfies this recurrence, then $T(n) = n \log_2 n$.

↑ assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Inductive hypothesis: $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n \log_2 n + 2n \\ &= 2n\left(\log_2 (2n) - 1\right) + 2n \\ &= 2n \log_2 (2n) \end{aligned}$$

Analysis of Mergesort Recurrence

Claim. If T(n) satisfies the following recurrence, then T(n) $\leq n \lceil \lg n \rceil$.

$\lg$ = $\log_2 n$

$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
- Induction step: assume true for 1, 2, ... , n–1.

$$\begin{aligned} T(n) &\leq T(n_1) + T(n_2) + n \\ &\leq n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &\leq n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &= n \lceil \lg n_2 \rceil + n \\ &\leq n(\lceil \lg n \rceil - 1) + n \\ &= n \lceil \lg n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\leq \left\lceil 2^{\lceil \lg n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \lg n \rceil} / 2 \\ \Rightarrow \lg n_2 &\leq \lceil \lg n \rceil - 1 \end{aligned}$$

Master Divide and Conquer Recurrence

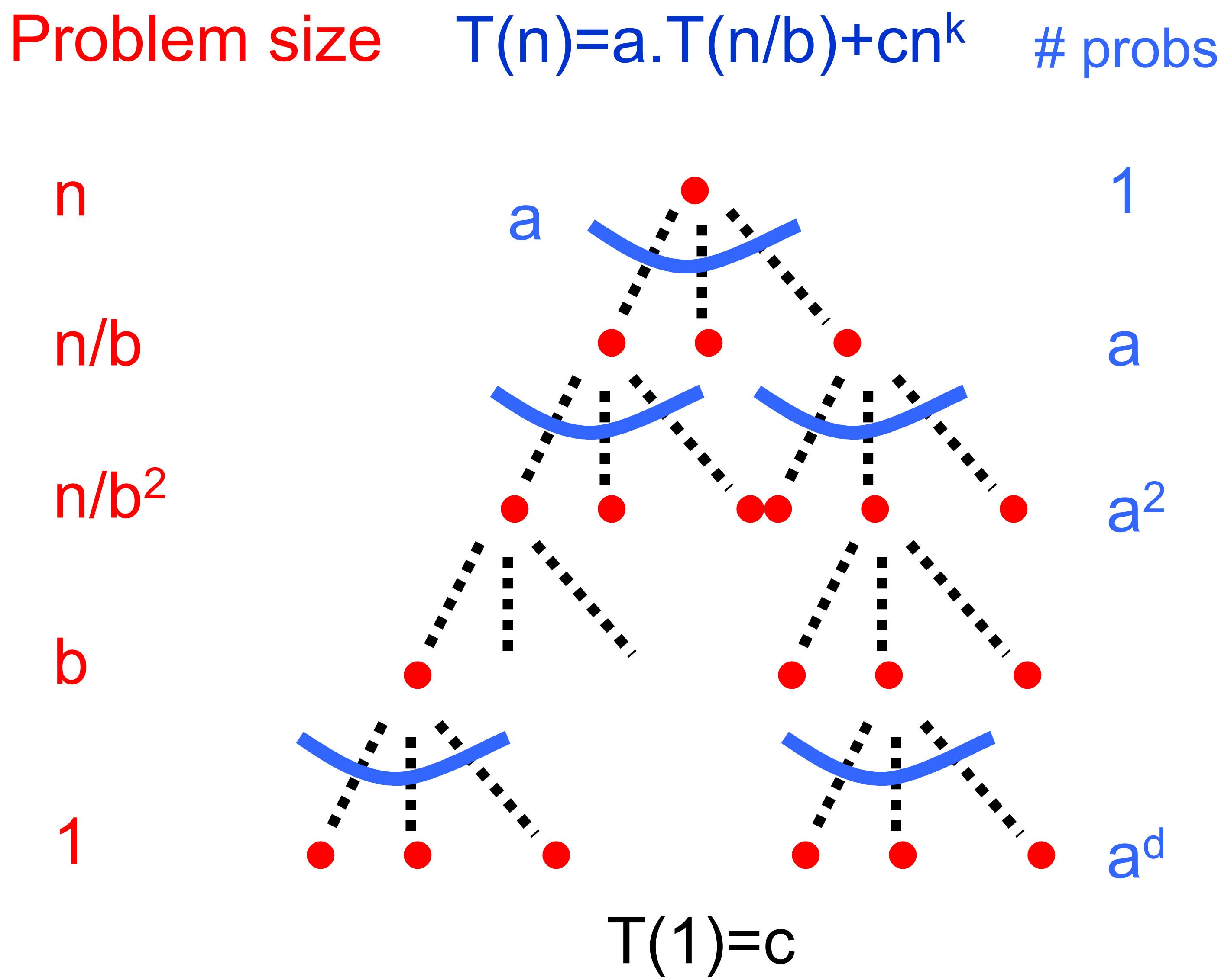
Let a and b be positive constants.

If $T(n) \leq a \cdot T(n/b) + c \cdot n^k$ for n > b then

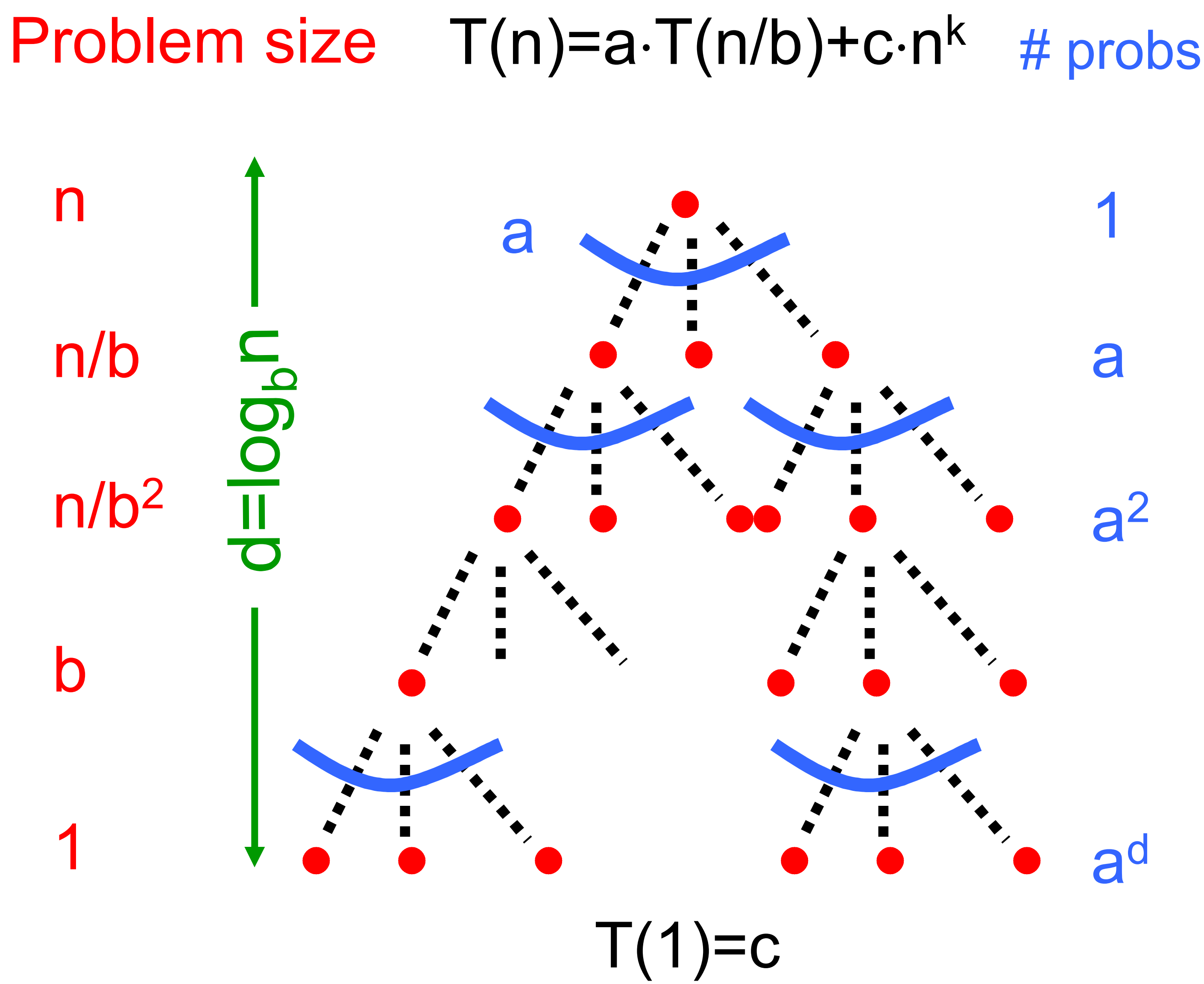
- if $a > b^k$ then T(n) is $\Theta(n^{\log_b a})$
- if $a < b^k$ then T(n) is $\Theta(n^k)$
- if $a = b^k$ then T(n) is $\Theta(n^k \log n)$

Works even if it is $\lceil n/b \rceil$ instead of n/b.

Proving Master recurrence

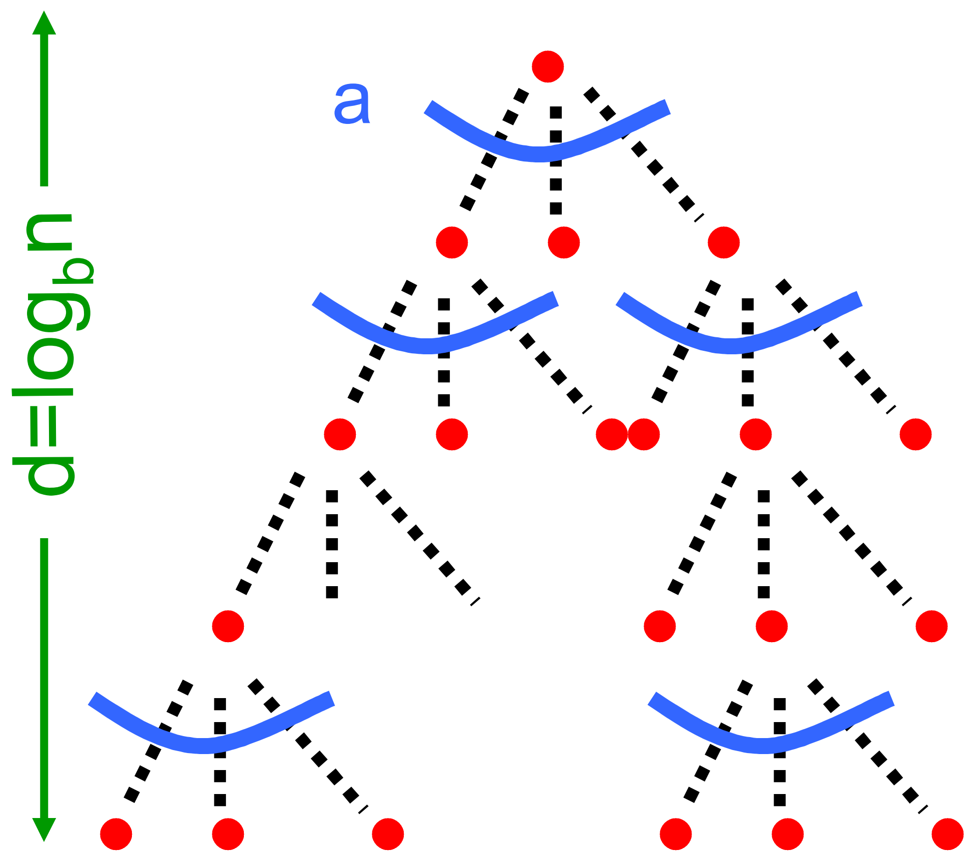


Proving Master recurrence



Proving Master recurrence

Problem size	$T(n)=a\cdot T(n/b)+c\cdot n^k$	# probs	cost
n		1	cn^k
n/b		a	$c\cdot a\cdot n^k/b^k$
n/b^2		a^2	$c\cdot a^2\cdot n^k/b^{2k}$ $=c\cdot n^k(a/b^k)^2$
b			
1		a^d	$c\cdot n^k(a/b^k)^d$ $=c\cdot a^d$



$T(1)=c$

Geometric Series

$$S \quad = \; t + \; tr \; + tr^2 + \ldots + tr^{n-1}$$

$$r \cdot S \quad = \qquad tr \; + tr^2 + \ldots + tr^{n-1} + tr^n$$

$$(r-1)S = tr^n - t$$

so $S = t\,(r^n - 1)/(r-1)$ if $r \neq 1$.

Simple rule

- If $r \neq 1$ then S is a constant times the largest term in series

Total Cost

Geometric series

- ratio a/b^k
- $d+1 = \log_b n + 1$ terms
- first term cn^k, last term ca^d

If $a/b^k=1$

- all terms are equal T(n) is $\Theta(n^k \log n)$

If $a/b^k<1$

- first term is largest T(n) is $\Theta(n^k)$

If $a/b^k>1$

- last term is largest T(n) is $\Theta(a^d) = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a})$

  (To see this take $\log_b$ of both sides)

13.5 Median Finding and Quicksort

Order problems: Find the k^{th} largest

Runtime models

- Machine Instructions
- Comparisons

Maximum

- **O(n)** time
- **n-1** comparisons

2^{nd} Largest

- **O(n)** time
- ? Comparisons

k^{th} largest for k = n/2

- Easily done in O(n log n) time with sorting
- How can the problem be solved in O(n) time?

QuickSelect(k, n) – find the k-th largest from a list of length n

Annoucements

- Homework 4 will be out later today, due date in 2 weeks on Wednesday 2/15
- The midterm is next Wednesday 2/8/2012
- Divide and conquer is not included in the midterm but recurrences are included.
- We will post sample exercises for recurrences on the webpage along with their solutions for practice.
- Remember NO outside sources (Google, other textbooks, people not in the class, etc.) may not be consulted on the homework

Divide and Conquer

Linear time solution: $T(n) = n + T(\alpha n)$ for $\alpha < 1$

QuickSelect algorithm – in linear time, reduce the problem from selecting the k-th largest of n to the j-th largest of αn, for $\alpha < 1$

```
QSelect(k, S)
    Choose element x from S
    S_L = {y in S | y < x }
    S_E = {y in S | y = x }
    S_G = {y in S | y > x }
    if |S_L| ≥ k
        return QSelect(k, S_L)
    else if |S_L| + |S_E| ≥ k
        return y in S_E
    else
        return QSelect(k - |S_L| - |S_E|, S_G)
```

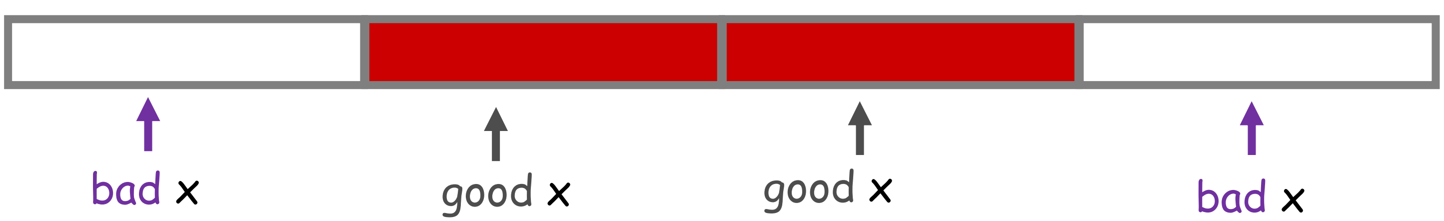
"Choose an element x": Random Selection

Ideally, we would choose an x in the middle, to reduce both sets in half and guarantee progress. But it's enough to choose x at random

Consider a call to QSelect(k, S), and let S' be the elements passed to the recursive call.

With probability at least $\frac{1}{2}$, $|S'| < \frac{3}{4}|S|$

$\Rightarrow$ On average only 2 recursive calls before the size of S' is at most 3n/4



elements of S listed in sorted order

Expected runtime is O(n)

Given x, one pass over S to determine S_L, S_E, and S_G and their sizes: cn time.

- Expect 2cn cost before size of S′ drops to at most 3|S|/4

Let T(n) be the expected running time: $T(n) \le T(3n/4) + 2cn$

By Master's Theorem, $T(n) = O(n)$

Making the algorithm deterministic

- In O(n) time, find an element that guarantees that the larger set in the split has size at most $\frac{3}{4}$ n
- BFPRT (Blum-Floyd-Pratt-Rivest-Tarjan) Algorithm

Quicksort

Sorting. Given a set of n distinct elements S, rearrange them in ascending order.

```
RandomizedQuicksort(S) {
   if |S| = 0 return

   choose a splitter aᵢ ∈ S uniformly at random
   foreach (a ∈ S) {
      if      (a < aᵢ) put a in S⁻
      else if (a > aᵢ) put a in S⁺
   }
   RandomizedQuicksort(S⁻)
   output aᵢ
   RandomizedQuicksort(S⁺)
}
```

Remark. Can implement in-place.

O(log n) extra space

Quicksort

Running time.

- [Best case.] Select the median element as the splitter: quicksort makes $\Theta(n \log n)$ comparisons.
- [Worst case.] Select the smallest element as the splitter: quicksort makes $\Theta(n^2)$ comparisons.

Randomize. Protect against worst case by choosing splitter at random.

Intuition. If we always select an element that is bigger than 25% of the elements and smaller than 25% of the elements, then quicksort makes $\Theta(n \log n)$ comparisons.

Notation. Label elements so that $x_1 < x_2 < \ldots < x_n$.

Expected run time for QuickSort: "Global analysis"

Count comparisons

a_i, a_j – elements in positions i and j in the final sorted list. p_{ij} the probability that a_i and a_j are compared

Expected number of comparisons: $\Sigma_{i<j}\, p_{ij}$

Prob a_i and a_j are compared:

- If a_i and a_j are compared then it must be during the call when they end up in different subproblems
    - Before that, they aren't compared to each other
    - After they aren't compared to each other
- During this step they are only compared if one of them is the pivot
- Since all elements between a_i and a_j are also in the subproblem this is 2 out of at least j-i+1 choices

Lemma: $P_{ij} \leq 2/(j - i + 1)$

Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is O(n log n).
Pf.

$$\sum_{1 \le i < j \le n} \frac{2}{j-i+1} = 2 \sum_{i=1}^{n} \sum_{j=2}^{i} \frac{1}{j} \le 2n \sum_{j=1}^{n} \frac{1}{j} \approx 2n \int_{x=1}^{n} \frac{1}{x} dx = 2n \ln n$$

probability that i and j are compared

Theorem. [Knuth 1973] Stddev of number of comparisons is ~ 0.65n.

Ex. If n = 1 million, the probability that randomized quicksort takes less than 4n ln n comparisons is at least 99.94%.

Chebyshev's inequality. $\Pr[|X - \mu| \ge k\delta] \le 1 / k^2$.

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.
- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

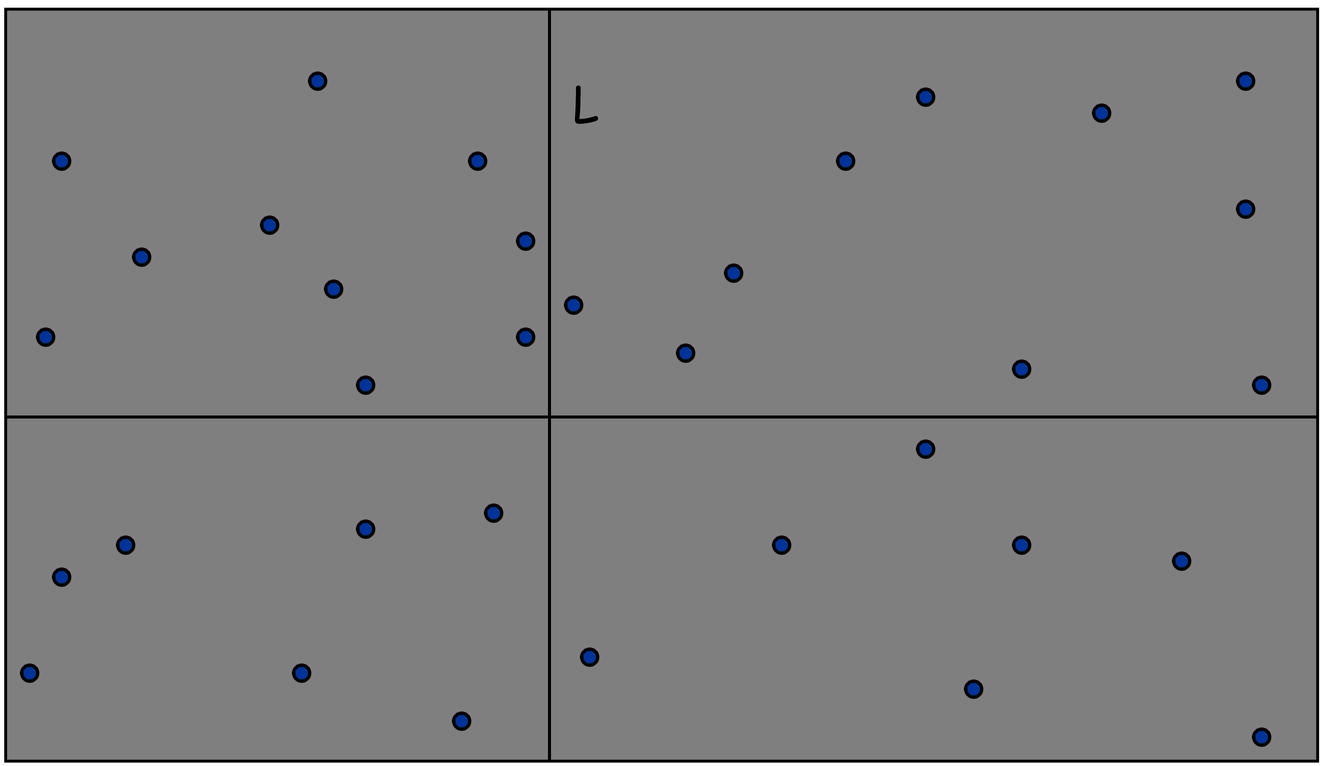
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

to make presentation cleaner

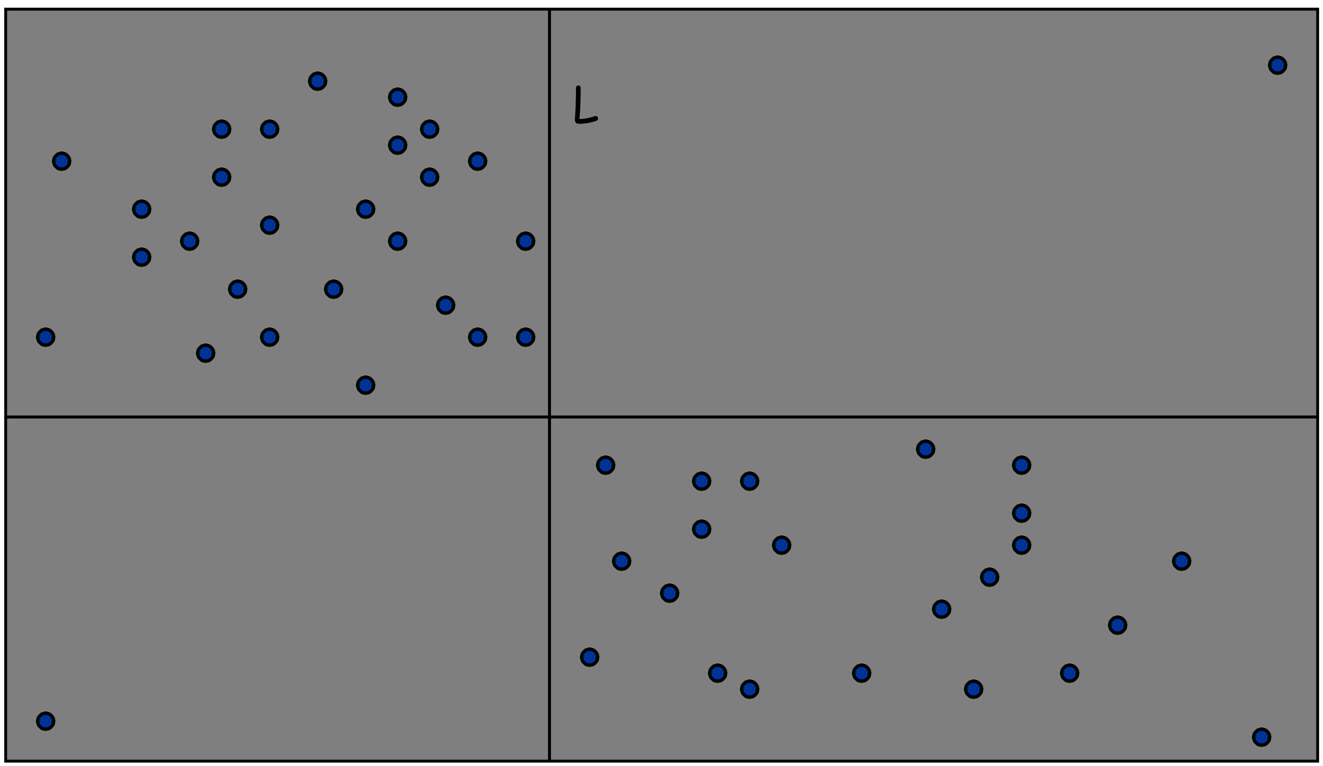
Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.



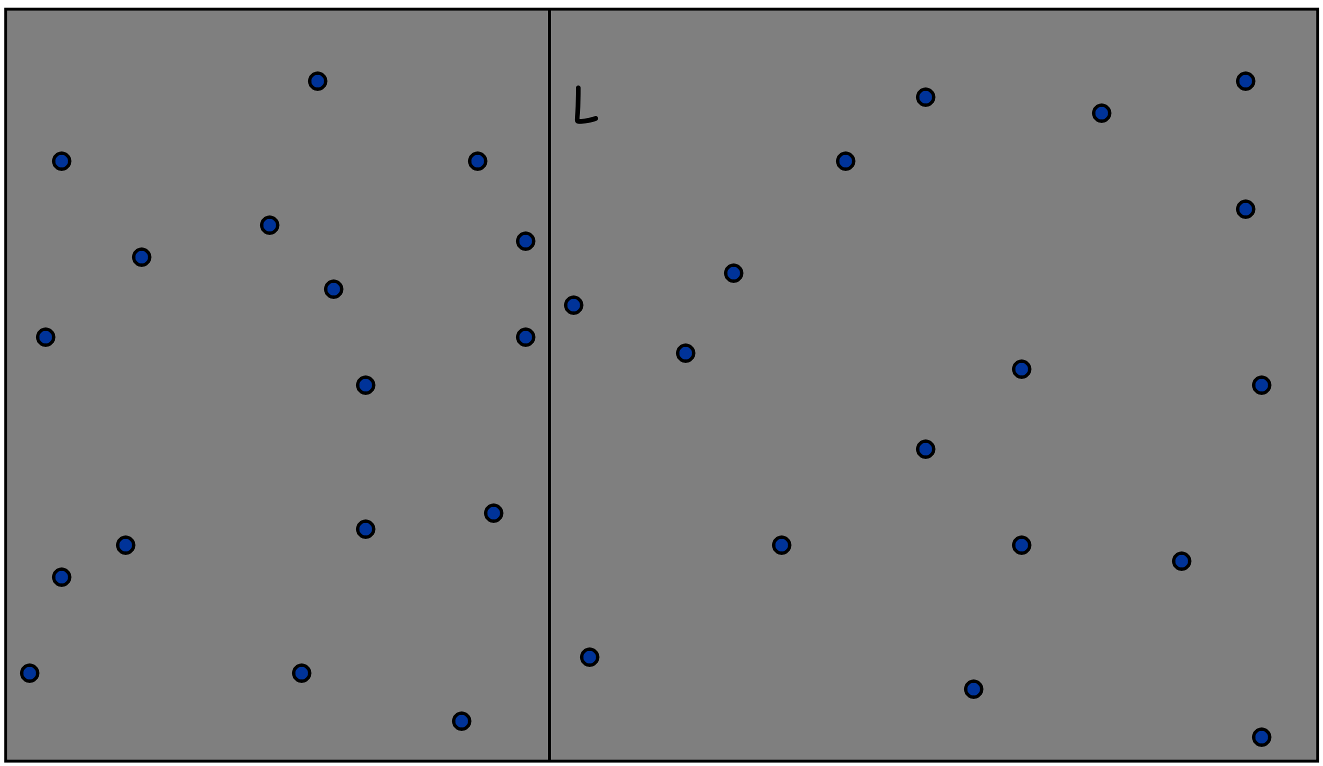
Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

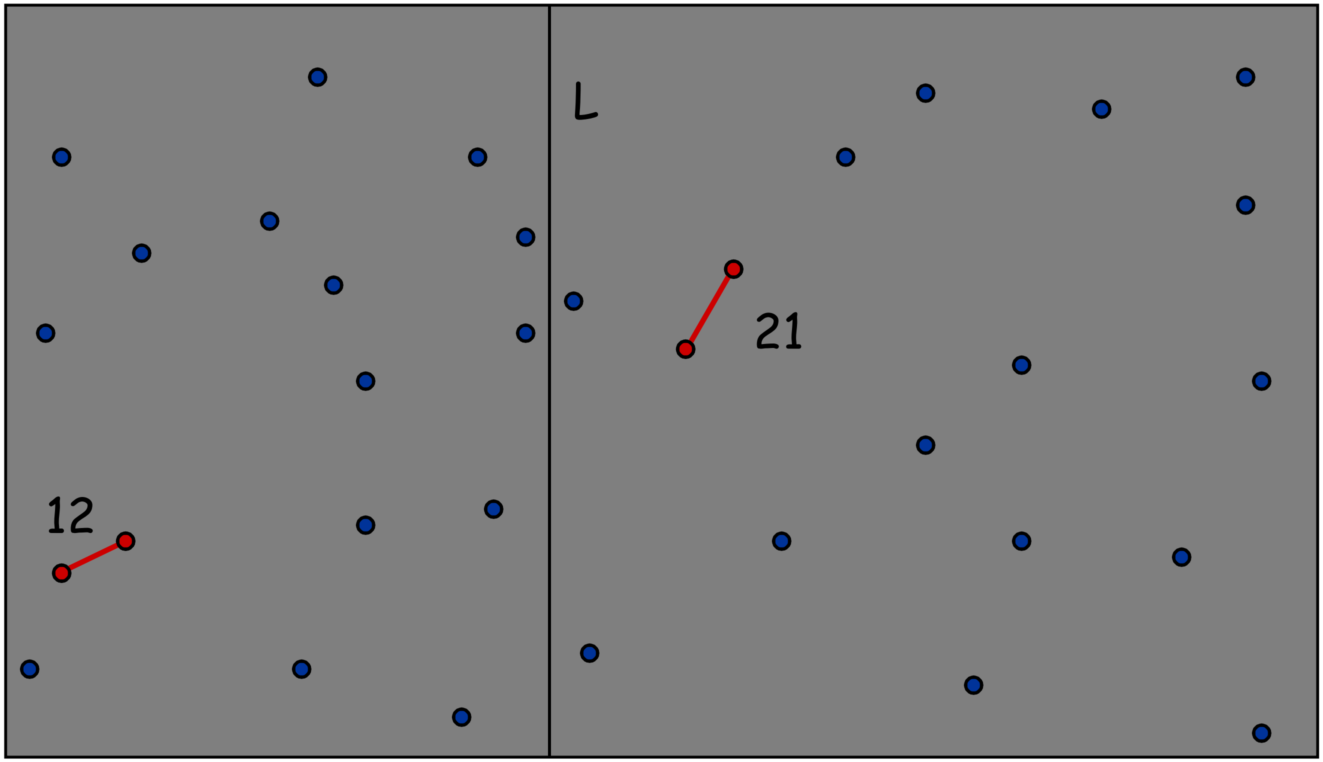
Algorithm.
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



Closest Pair of Points

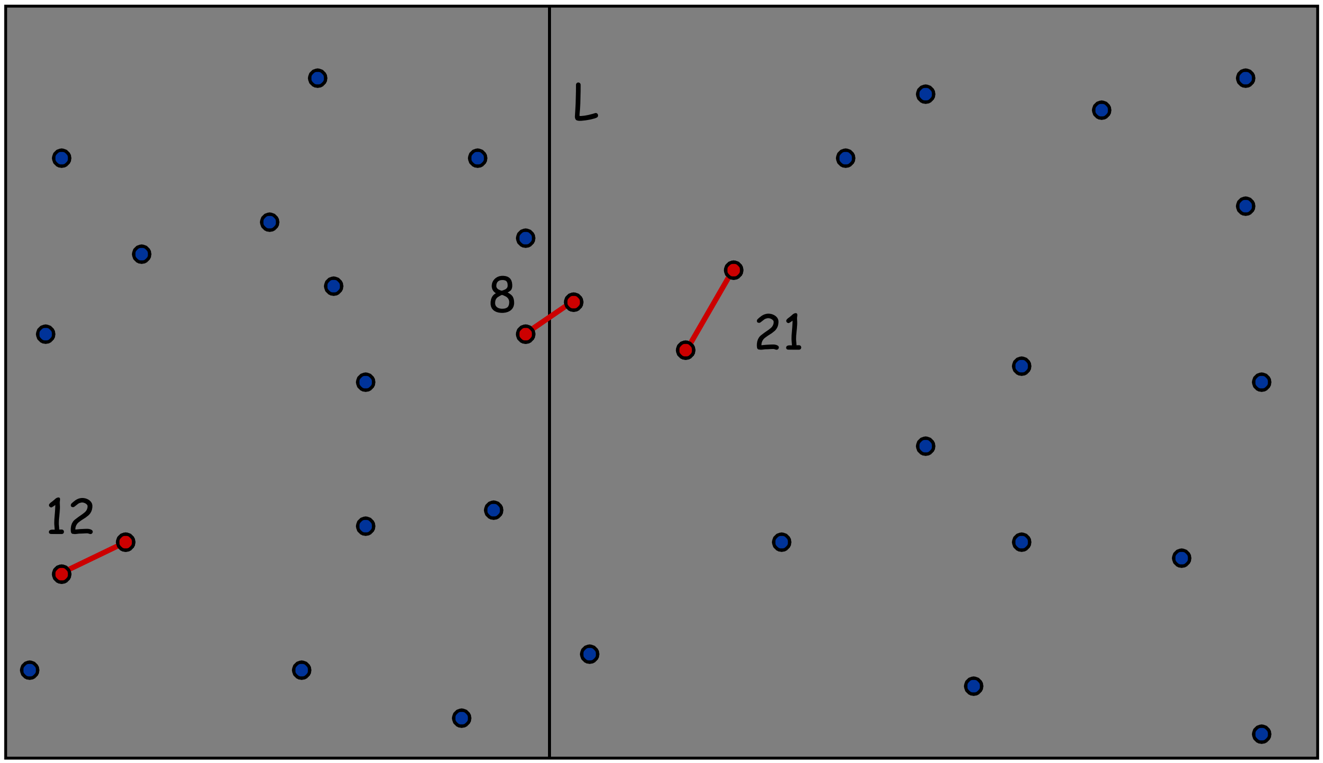
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.
- Conquer: find closest pair in each side recursively.



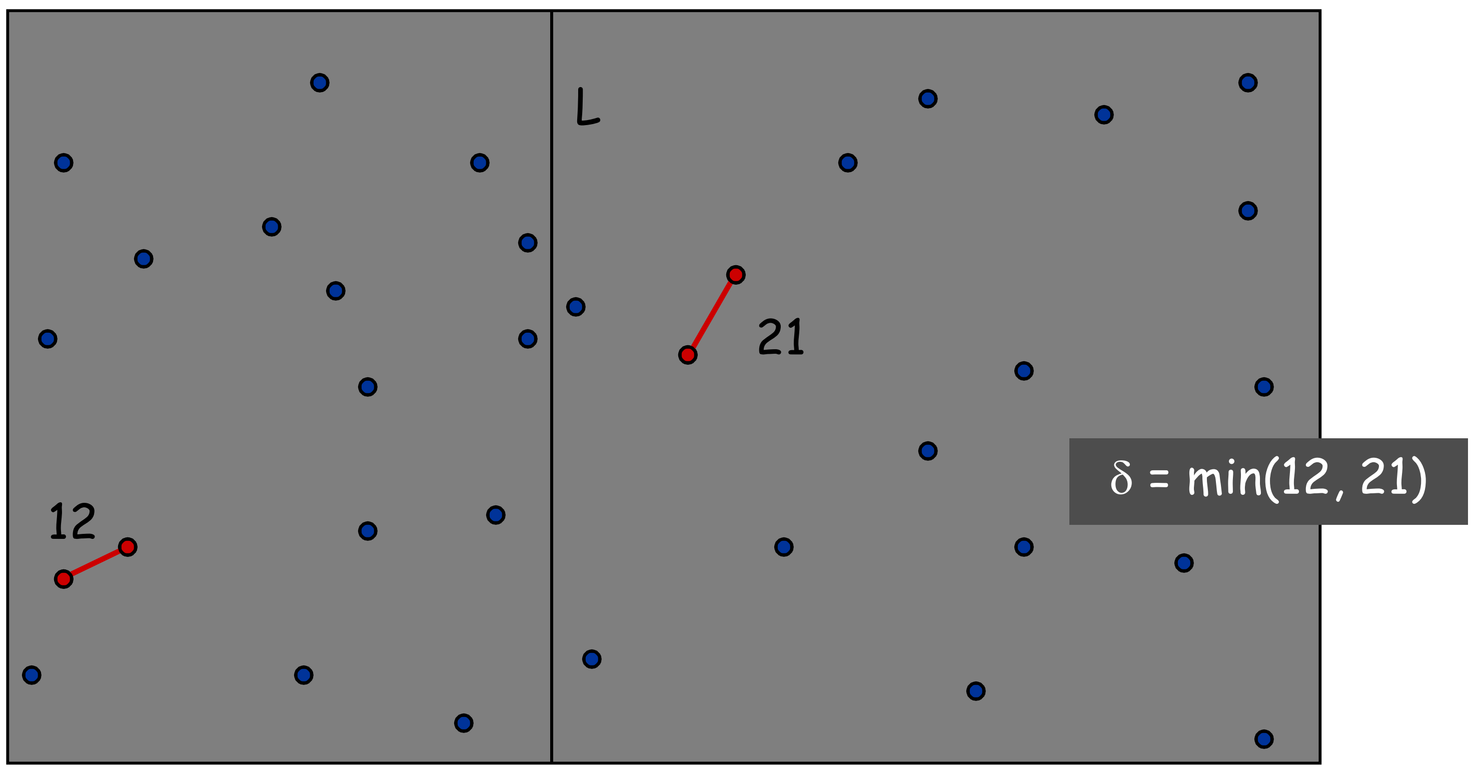
Closest Pair of Points

Algorithm.
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

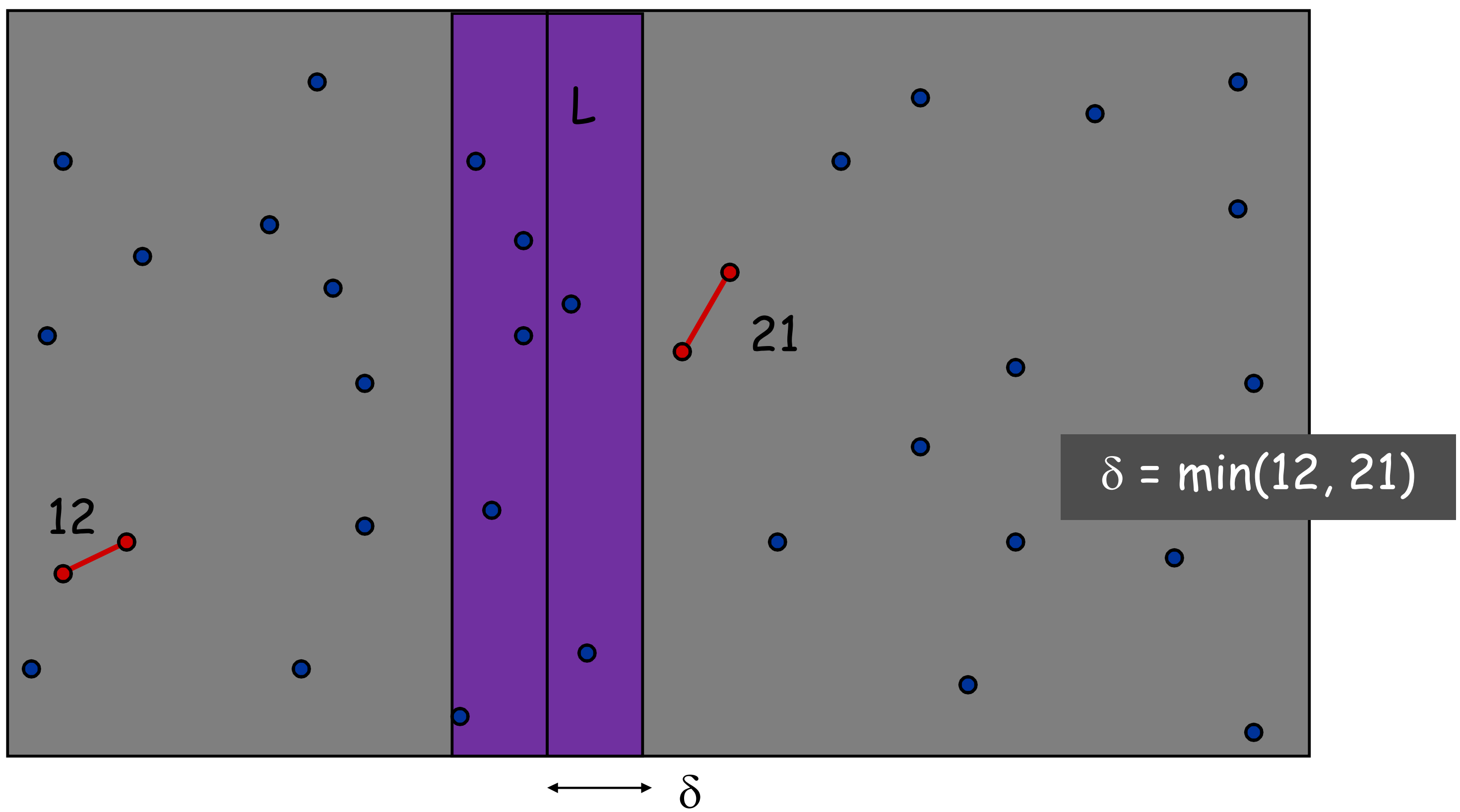
Find closest pair with one point in each side, assuming that distance < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

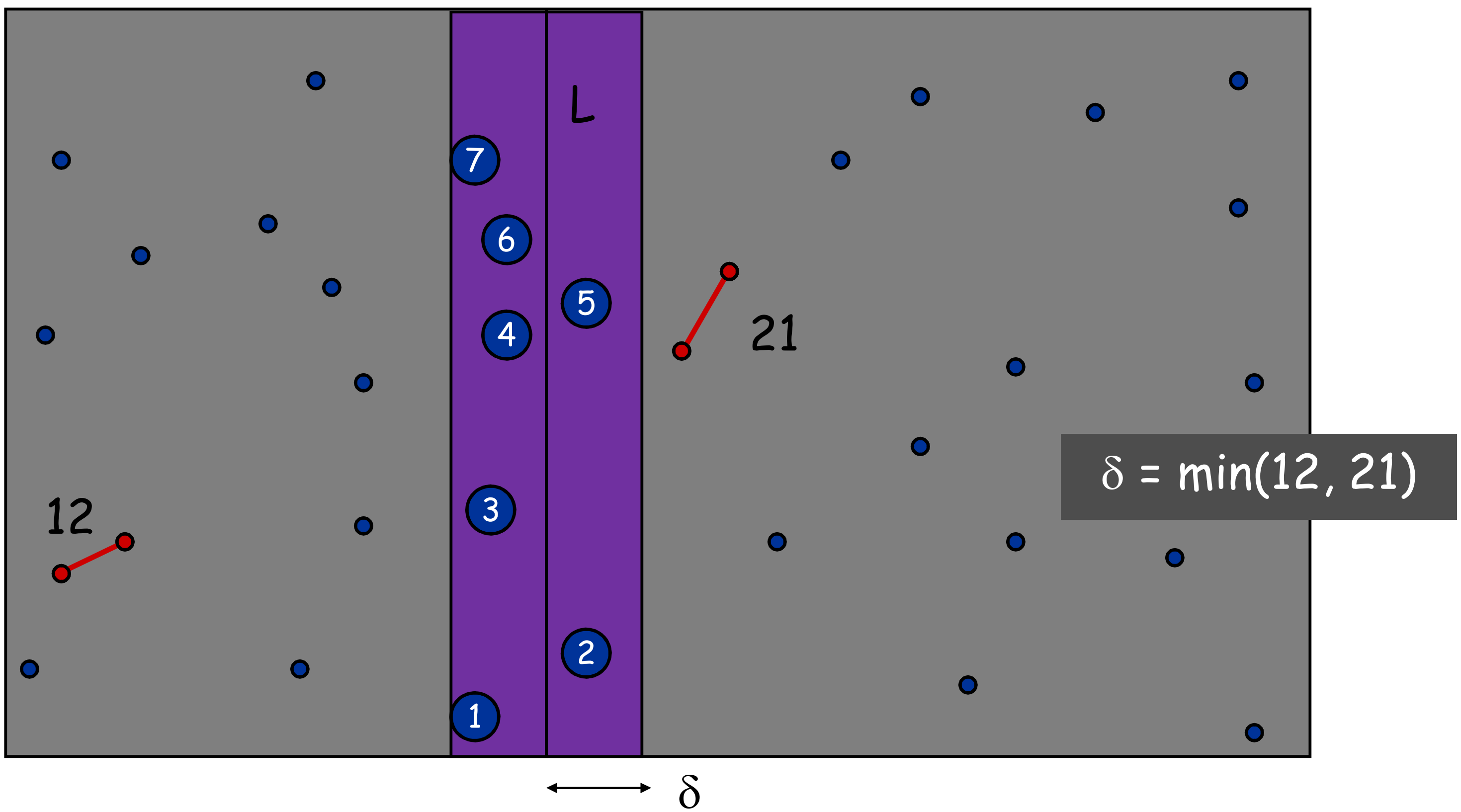
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

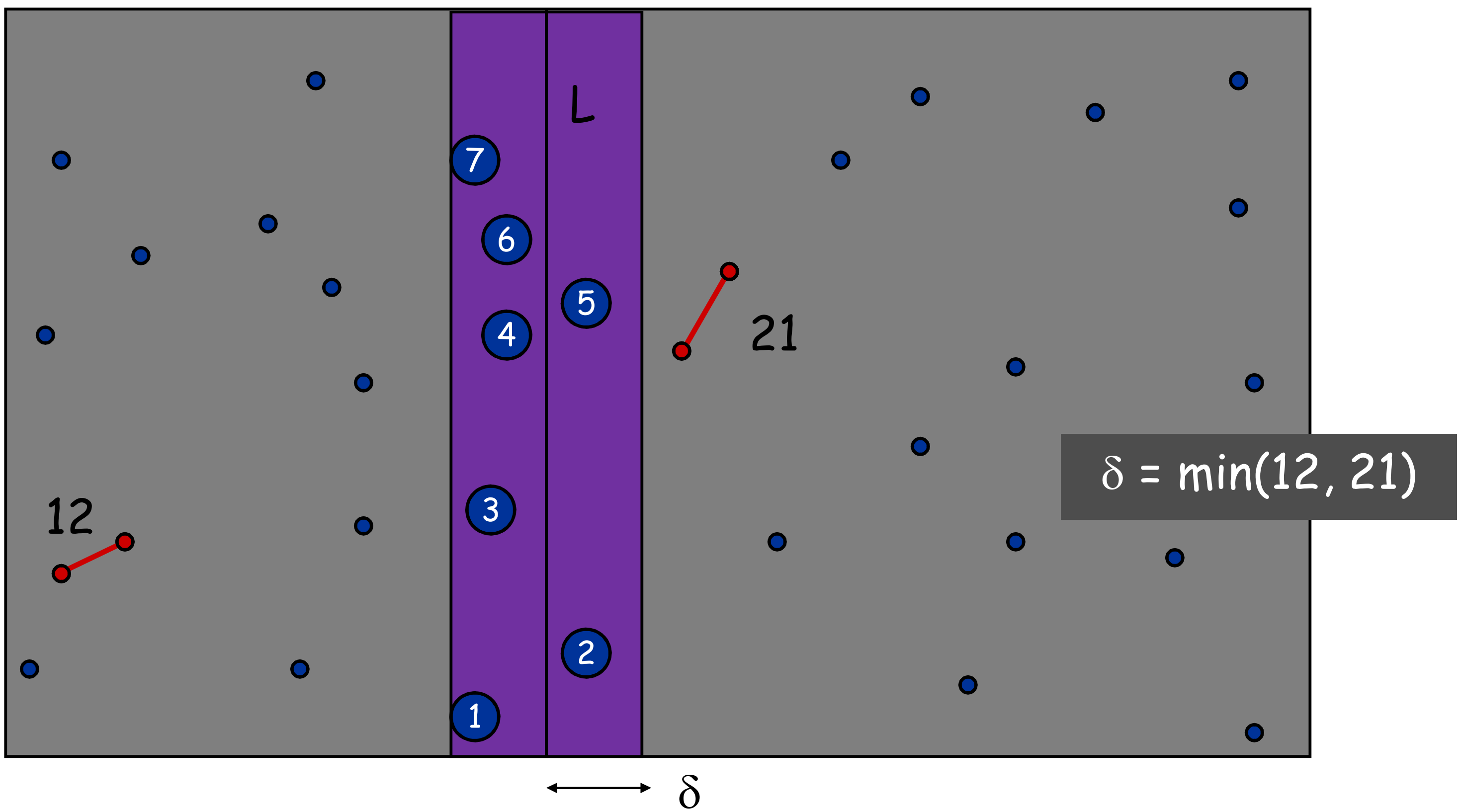
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



Closest Pair of Points

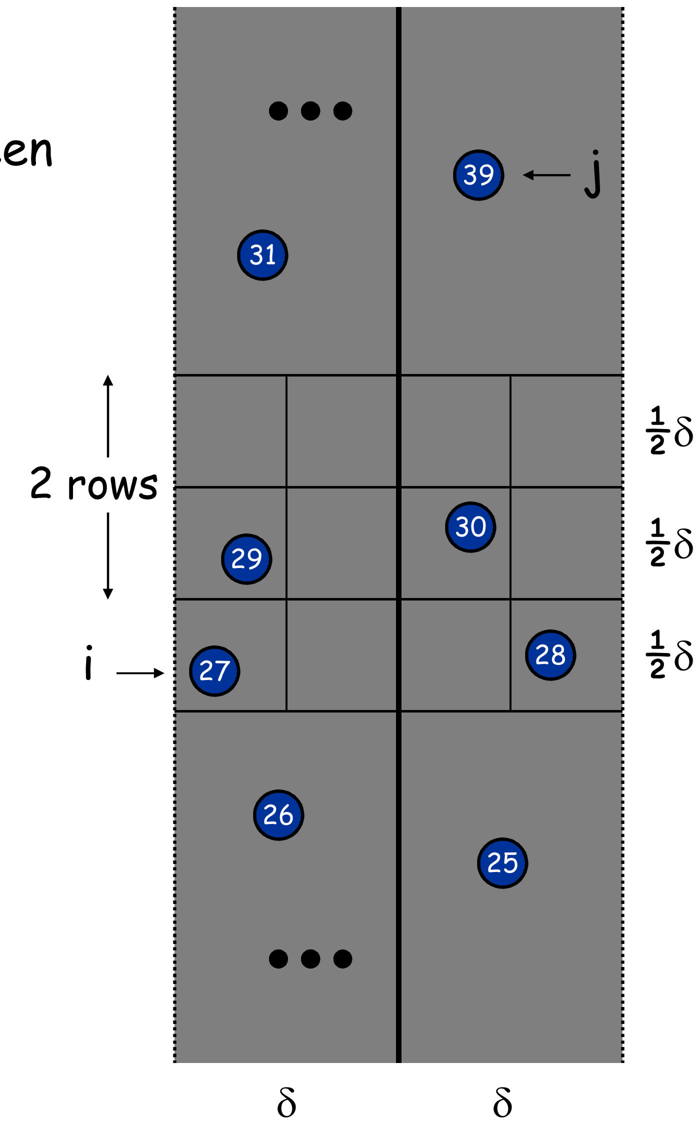
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.
- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Corollary For each point s_i, we only need to check its distance to the 11 points that precedes it in the y-coordinate order.

Fact. Still true if we replace 11 with 6.



Closest Pair Algorithm

```
Closest-Pair(p₁, …, pₙ) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

$O(n \log n)$

$2T(n / 2)$

$O(n)$

$O(n \log n)$

$O(n)$

Closest Pair of Points: Analysis

Running time.

$$T(n) \leq 2T\big(n/2\big) + O(n \log n) \quad \Rightarrow \quad T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \leq 2T\big(n/2\big) + O(n) \quad \Rightarrow \quad T(n) = O(n \log n)$$

5.5 Integer Multiplication

Integer Arithmetic

Add. Given two n-digit integers a and b, compute a + b.

- O(n) bit operations.

Multiply. Given two n-digit integers a and b, compute a × b.

- Brute force solution: $\Theta(n^2)$ bit operations.

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Add

```
                  1 1 0 1 0 1 0 1
                * 0 1 1 1 1 1 0 1
                -----------------
                1 1 0 1 0 1 0 1 0
              0 0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1 0
          1 1 0 1 0 1 0 1 0
        1 1 0 1 0 1 0 1 0
      1 1 0 1 0 1 0 1 0
    1 1 0 1 0 1 0 1 0
  0 0 0 0 0 0 0 0 0
---------------------------------
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1 0
```

Multiply

Multiplying Faster

If you analyze our usual grade school algorithm for multiplying numbers

- $\Theta(n^2)$ time
- On real machines each "digit" is, e.g., 32 bits long but still get $\Theta(n^2)$ running time with this algorithm when run on n-bit multiplication

We can do better!

- We'll describe the basic ideas by multiplying polynomials rather than integers
- Advantage is we don't get confused by worrying about carries at first

Notes on Polynomials

These are just formal sequences of coefficients

- when we show something multiplied by x^k it just means shifted k places to the left – basically no work

Usual polynomial multiplication

$$
\begin{array}{r}
4x^2 + 2x + 2 \\
x^2 - 3x + 1 \\
\hline
4x^2 + 2x + 2 \\
-12x^3 - 6x^2 - 6x \phantom{{}+2} \\
4x^4 + 2x^3 + 2x^2 \phantom{{}- 6x + 2} \\
\hline
4x^4 - 10x^3 + 0x^2 - 4x + 2
\end{array}
$$

Polynomial Multiplication

Given:

- Degree $n-1$ polynomials P and Q
  - $P = a_0 + a_1\, x + a_2\, x^2 + \ldots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$
  - $Q = b_0 + b_1\, x + b_2\, x^2 + \ldots + b_{n-2}x^{n-2} + b_{n-1}x^{n-1}$

Compute:

- Degree $2n-2$ Polynomial $P\,Q$
- $P\,Q = a_0b_0 + (a_0b_1+a_1b_0)\, x + (a_0b_2+a_1b_1+a_2b_0)\, x^2 + \ldots +$
  $(a_{n-2}b_{n-1}+a_{n-1}b_{n-2})\, x^{2n-3} + a_{n-1}b_{n-1}\, x^{2n-2}$

Obvious Algorithm:

- Compute all a_ib_j and collect terms
- $\Theta(n^2)$ time

Naive Divide and Conquer

Assume $n=2k$

- $P = (a_0 + a_1 \; x + a_2 \, x^2 + \ldots + a_{k-2} \, x^{k-2} + a_{k-1} \, x^{k-1}) +$
  $(a_k + a_{k+1} \, x + \qquad \ldots + a_{n-2} x^{k-2} + a_{n-1} x^{k-1}) \, x^k$

  $= P_0 + P_1 \, x^k$ where P_0 and P_1 are degree $k-1$ polynomials

- Similarly $Q = Q_0 + Q_1 \, x^k$

- $P \, Q = (P_0 + P_1 x^k)(Q_0 + Q_1 x^k) = P_0 Q_0 + (P_1 Q_0 + P_0 Q_1) x^k + P_1 Q_1 x^{2k}$

- 4 sub-problems of size $k=n/2$ plus linear combining

  $T(n) = 4 \cdot T(n/2) + cn$ Solution $T(n) = \Theta(n^2)$

Karatsuba's Algorithm

A better way to compute the terms

- Compute
  - $A \leftarrow P_0Q_0$
  - $B \leftarrow P_1Q_1$
  - $C \leftarrow (P_0+P_1)(Q_0+Q_1) = P_0Q_0+P_1Q_0+P_0Q_1+P_1Q_1$
- Then
  - $P_0Q_1+P_1Q_0 = C-A-B$
  - So $PQ=A+(C-A-B)x^k+Bx^{2k}$
- 3 sub-problems of size $n/2$ plus $O(n)$ work
  - $T(n) = 3\ T(n/2) + cn$
  - $T(n) = O(n^\alpha)$ where $\alpha = \log_2 3 = 1.59\ldots$

Karatsuba's algorithm and evaluation and interpolation

Karatsuba's algorithm can be thought of as a way of multiplying degree 1 polynomials (which have 2 coefficients) using fewer multiplications

- $PQ=(P_0+P_1z)(Q_0+Q_1z)$
  $= P_0Q_0 + (P_1Q_0+P_0Q_1)z + P_1Q_1z^2$
- Evaluate at 0,1,-1 (Could also use other points)
  - $A = P(0)Q(0)= P_0Q_0$
  - $C = P(1)Q(1)=(P_0+P_1)(Q_0+Q_1)$
  - $D = P(-1)Q(-1)=(P_0 -P_1)(Q_0 -Q_1)$

Multiplication

Polynomials

- Naïve: $\Theta(n^2)$
- Karatsuba: $\Theta(n^{1.59\ldots})$
- Best known: $\Theta(n \log n)$
  - "Fast Fourier Transform"
  - FFT widely used for signal processing

Integers

- Similar, but some ugly details re: carries, etc. gives $\Theta(n \log n \log\log n)$,
  - mostly unused in practice except for symbolic manipulation systems like Maple

Matrix Multiplication

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

```
for i=1 to n
    for j=1 to n
        C[i,j]←0
        for k=1 to n
            C[i,j]=C[i,j]+A[i,k]·B[k,j]
        endfor
    endfor
endfor
```

n^3 multiplications, n^3-n^2 additions

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right] \bullet \left[\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ \hline b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{array}\right]$$

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \bullet \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ \hline a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{array}\right]$$

$$= \left[\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ \hline A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right]$$

Simple Divide and Conquer

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}$$

$T(n) = 8T(n/2) + 4(n/2)^2 = 8T(n/2) + n^2$

- $8 > 2^2$ so $T(n)$ is $\Theta(n^{\log_b a}) = \Theta(n^{\log_2 8}) = \Theta(n^3)$

Strassen's Divide and Conquer Algorithm

Strassen's algorithm

- Multiply 2x2 matrices using 7 instead of 8 multiplications (and lots more than 4 additions)

- $T(n) = 7\,T(n/2) + cn^2$
  - $7 > 2^2$ so $T(n)$ is $\Theta(n^{\log_2 7})$ which is $O(n^{2.81\ldots})$

- Fastest algorithms theoretically use $O(n^{2.373})$ time
  - not practical but Strassen's is practical provided calculations are exact and we stop recursion when matrix has size about 100 (maybe 10)

The algorithm

$P_1 \leftarrow A_{12}(B_{11}+B_{21})$; $P_2 \leftarrow A_{21}(B_{12}+B_{22})$

$P_3 \leftarrow (A_{11} - A_{12})B_{11}$; $P_4 \leftarrow (A_{22} - A_{21})B_{22}$

$P_5 \leftarrow (A_{22} - A_{12})(B_{21} - B_{22})$

$P_6 \leftarrow (A_{11} - A_{21})(B_{12} - B_{11})$

$P_7 \leftarrow (A_{21} - A_{12})(B_{11}+B_{22})$

7 multiplications.
18 = 10 + 8 additions (or subtractions).

$C_{11} \leftarrow P_1+P_3$; $C_{12} \leftarrow P_2+P_3+P_6 - P_7$

$C_{21} \leftarrow P_1+P_4+P_5+P_7$; $C_{22} \leftarrow P_2+P_4$

Fast Matrix Multiplication in Practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.
- Crossover to classical algorithm around n = 128.

Common misperception: "Strassen is only a theoretical curiosity."

- Advanced Computation Group at Apple Computer reports 8x speedup on G4 Velocity Engine when n ~ 2,500.
- Range of instances where it's useful is a subject of controversy.

Remark. Can "Strassenize" Ax=b, determinant, eigenvalues, and other matrix ops.

Fast Matrix Multiplication in Theory

Q. Multiply two 2-by-2 matrices with only 7 scalar multiplications?
A. Yes! [Strassen, 1969] $\Theta(n^{\log_2 7}) = O(n^{2.81})$

Q. Multiply two 2-by-2 matrices with only 6 scalar multiplications?
A. Impossible. [Hopcroft and Kerr, 1971] $\Theta(n^{\log_2 6}) = O(n^{2.59})$

Q. Two 3-by-3 matrices with only 21 scalar multiplications?
A. Also impossible. $\Theta(n^{\log_3 21}) = O(n^{2.77})$

Decimal wars.

- December, 1979: $O(n^{2.521813})$.
- January, 1980: $O(n^{2.521801})$.

Fast Matrix Multiplication in Theory

Until Oct 2011. $O(n^{2.376})$ [Coppersmith-Winograd, 1987.]

Best known. $O(n^{2.373})$ [V . Williams, Nov 2011]

Conjecture. $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

Caveat. not practical but Strassen's is practical provided calculations are exact and we stop recursion when matrix has size about 100 (maybe 10)