# CSE 417: Algorithms and Computational Complexity

Winter 2009

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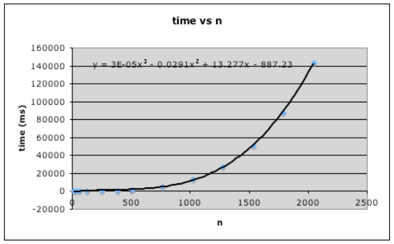
## Divide and Conquer Algorithms

## HW4 – Empirical Run Times

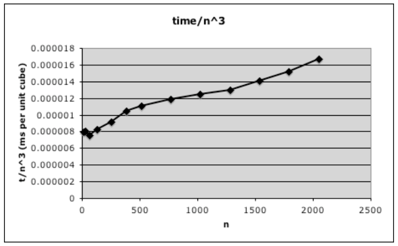



Plot Time vs n

Fit curve to it (e.g., with Excel)

Note: Higher degree polynomials fit better…

Plotting Time/(growth rate) vs n may be more sensitive – should be flat, but small n may be unrepresentative of asymptotics






## The Divide and Conquer Paradigm

Outline:

- General Idea
- Review of Merge Sort
- Why does it work?
  - Importance of balance
  - Importance of super-linear growth
- Some interesting applications
  - Closest points
  - Integer Multiplication
- Finding & Solving Recurrences



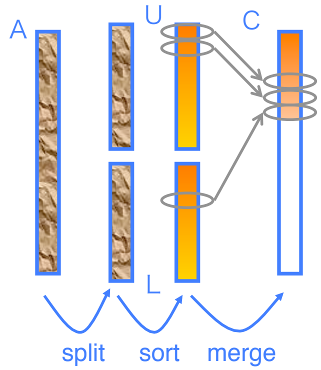
## Algorithm Design Techniques

Divide & Conquer

- Reduce problem to one or more sub-problems of the same type
- Typically, each sub-problem is at most a constant fraction of the size of the original problem
  - e.g. Mergesort, Binary Search, Strassen's Algorithm, Quicksort (kind of)

# Merge Sort

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A[1];
    New U:array[1:n/2] = MS(A[1..n/2]);
    New L:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(U,L));
    }
Merge(U,L: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n
       C[i] = "smaller of U[a], L[b] and correspondingly a++ or b++";
    Return C;
    }
```





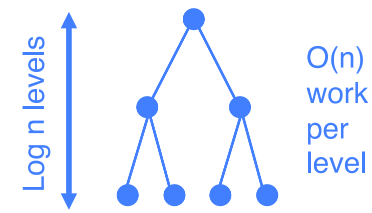

# Mergesort (review)

Mergesort: (recursively) sort 2 half-lists, then merge results.

$T(n) = 2T(n/2)+cn, \; n \geq 2$

$T(1) = 0$

Solution: $O(n \log n)$ (details later)






# Why Balanced Subdivision?

Alternative "divide & conquer" algorithm:

Sort n-1

Sort last 1

Merge them

$T(n) = T(n-1)+T(1)+3n \quad \text{for } n \geq 2$

$T(1) = 0$

Solution: $3n + 3(n-1) + 3(n-2) \ldots = \Theta(n^2)$



# Another D&C Approach

Suppose we've already invented DumbSort, taking time $n^2$

Try *Just One Level* of divide & conquer:

- DumbSort(first n/2 elements)
- DumbSort(last n/2 elements)
- Merge results

Time: $2\,(n/2)^2 + n = n^2/2 + n << n^2$

Almost twice as fast!

D&C in a nutshell

## Another D&C Approach, cont.

Moral 1: “two halves are better than a whole”

Two problems of half size are *better* than one full-size problem, even given the O(n) overhead of recombining, since the base algorithm has *super-linear* complexity.

Moral 2: “If a little's good, then more's better”

two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing. Best is usually full recursion down to some small constant size (balancing "work" vs "overhead").



## Another D&C Approach, cont.

Moral 3: unbalanced division less good:

$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$

The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can’t get 50-50 split, but balanced is better if you can.

This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

$(1)^2 + (n-1)^2 + n = n^2 - 2n + 2 + n$

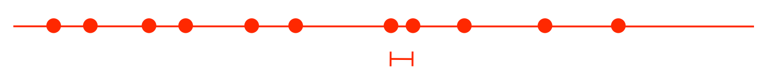
Little improvement here.



# 5.4 Closest Pair of Points

## Closest pair of points: 1 Dimensional Version

Given n points on the real line, find the closest pair



Closest pair is adjacent in ordered list

Time O(n log n) to sort, if needed

Plus O(n) to scan adjacent pairs

## Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.
- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

↑ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

1-D version. O(n log n) easy if points are on a line.

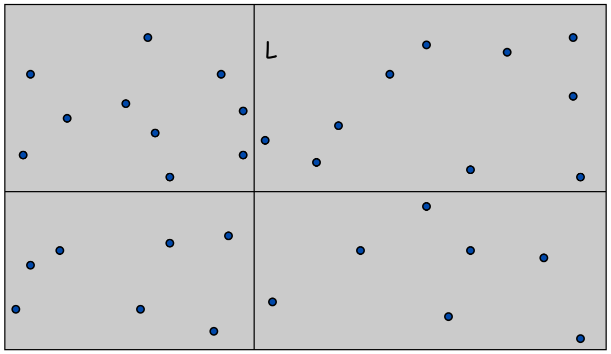
Assumption. No two points have same x coordinate.

↑ to make presentation cleaner



## Closest Pair of Points: First Attempt

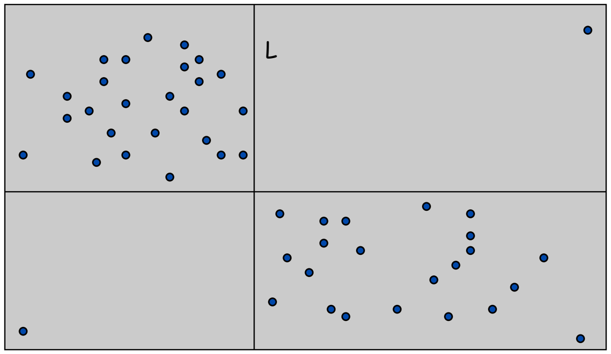
Divide. Sub-divide region into 4 quadrants.






## Closest Pair of Points: First Attempt

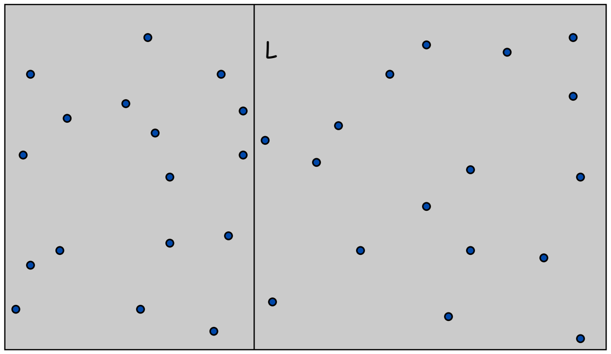
Divide. Sub-divide region into 4 quadrants.
Obstacle. Impossible to ensure n/4 points in each piece.






## Closest Pair of Points

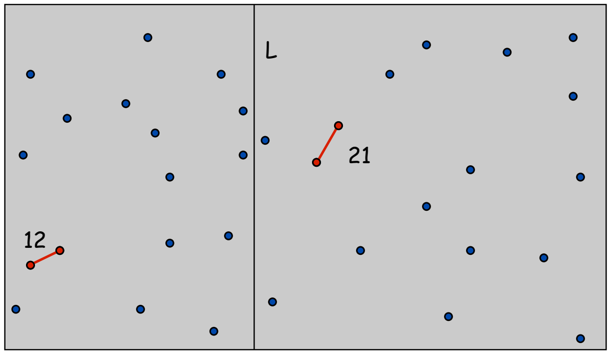
Algorithm.
- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.



## Closest Pair of Points

Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.

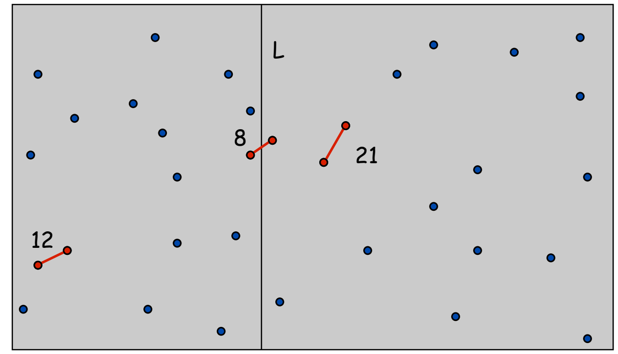





## Closest Pair of Points

Algorithm.

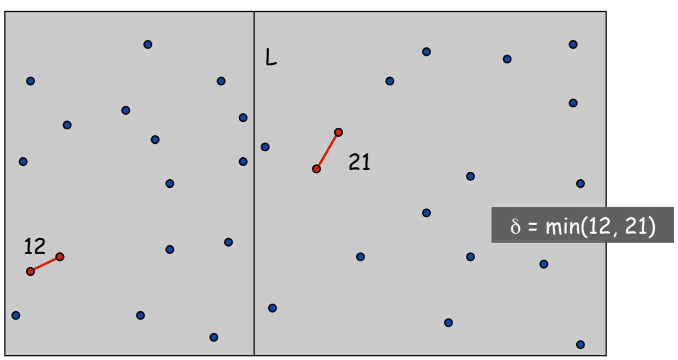
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.






## Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < $\delta$.

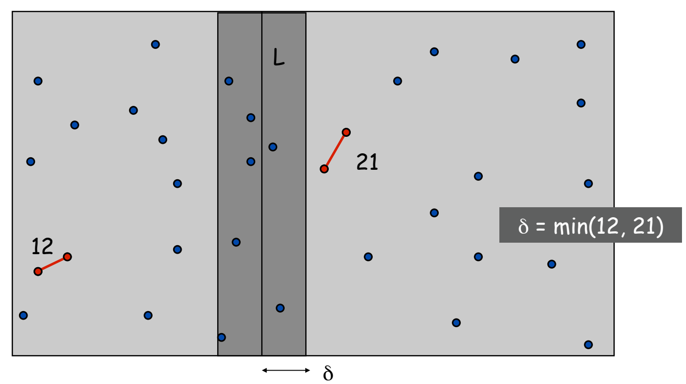





## Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < $\delta$.

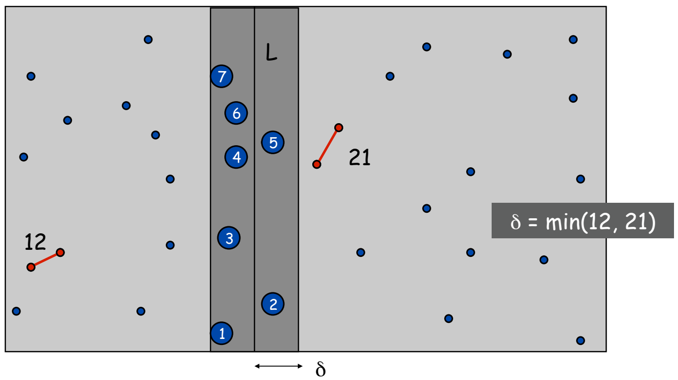
- Observation: only need to consider points within $\delta$ of line L.



## Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < $\delta$.

- Observation: only need to consider points within $\delta$ of line L.
- Sort points in $2\delta$-strip by their y coordinate.

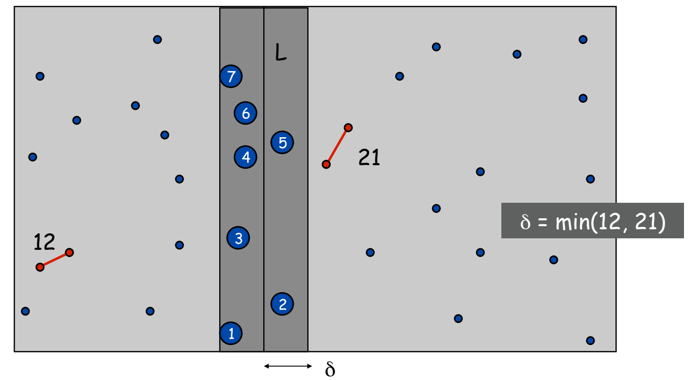





## Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < $\delta$.

- Observation: only need to consider points within $\delta$ of line L.
- Sort points in $2\delta$-strip by their y coordinate.
- Only check distances of those within 8 positions in sorted list!





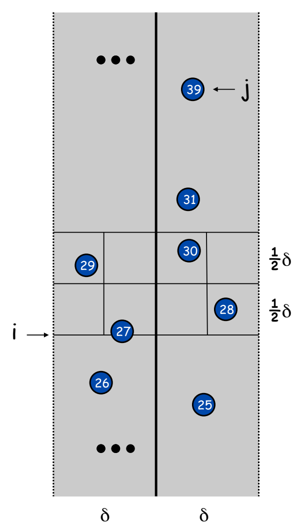

## Closest Pair of Points

Def. Let $s_i$ be the point in the $2\delta$-strip, with the $i^{th}$ smallest y-coordinate.

Claim. If $|i - j| > 8$, then the distance between $s_i$ and $s_j$ is $> \delta$.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- only 8 boxes






## Closest Pair Algorithm

```
Closest-Pair(p1, …, pn) {
   if(n <= ??) return ??

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
         δ = min(δ, distance between p[i] and p[i+k]);
         k++;

   return δ.
}
```

## Going From Code to Recurrence

Carefully define what you're counting, and write it down!

"Let C(n) be the number of comparisons between sort keys used by MergeSort when sorting a list of length n ≥ 1"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted.*

Write Recurrence(s)



## Merge Sort

```
MS(A: array[1..n]) returns array[1..n] {
  If(n=1) return A[1];
    New L:array[1:n/2] = MS(A[1..n/2]);
    New R:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(L,R));
  }
Merge(A,B: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n {
      C[i] = "smaller of A[a], B[b] and a++ or b++";
    Return C;
  }
```

Base Case

Recursive calls

Recursive case

Operations being counted



## The Recurrence

$$C(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2C(n/2) + (n-1) & \text{if } n > 1 \end{cases}$$

Base case

Recursive calls

One compare per element added to merged list, except the last.

Total time: proportional to C(n)

(loops, copying data, parameter passing, etc.)



## Going From Code to Recurrence

Carefully define what you're counting, and write it down!

"Let D(n) be the number of pairwise distance comparisons in the Closest-Pair Algorithm when run on n ≥ 1 points"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted*.

Write Recurrence(s)

## Closest Pair Algorithm

Base Case

Basic operations: distance calcs

Recursive calls (2)

Basic operations at this recursive level

```
Closest-Pair(p1, …, pn) {
   if(n <= 1) return ∞                                          0

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)                                 2D(n / 2)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ                   O(n)
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```



## Closest Pair of Points: Analysis

Running time.

$$D(n) \le \begin{cases} 0 & n = 1 \\ 2D(n/2) + 7n & n > 1 \end{cases} \Rightarrow D(n) = O(n \log n)$$

BUT - that's only the number of *distance calculations*

What if we counted comparisons?



## Closest Pair Algorithm

Base Case

Basic operations: *comparisons*

Recursive calls (2)

Basic operations at this recursive level

```
Closest-Pair(p1, …, pn) {
   if(n <= 1) return ∞                                          0

   Compute separation line L such that half the points          O(n log n)
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)                                 2C(n / 2)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)                                             1

   Delete all points further than δ from separation line L     O(n)

   Sort remaining points p[1]…p[m] by y-coordinate.             O(n log n)

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ                   O(n)
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```



## Closest Pair of Points: Analysis

Running time.

$$C(n) \le \begin{cases} 0 & n = 1 \\ 2C(n/2) + O(n \log n) & n > 1 \end{cases} \Rightarrow C(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points from scratch each time.

- Sort by x at top level only.
- Each recursive call returns δ and list of all points sorted by y
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

# 5.5 Integer Multiplication

## Integer Arithmetic

Add. Given two n-digit integers a and b, compute a + b.

- O(n) bit operations.

Multiply. Given two n-digit integers a and b, compute a × b.

- The "grade school" method: $\Theta(n^2)$ bit operations.

Add

```
1 1 1 1 1 1 0 1
  1 1 0 1 0 1 0 1
+ 0 1 1 1 1 1 0 1
-----------------
1 0 1 0 1 0 0 1 0
```

Multiply

```
                1 1 0 1 0 1 0 1
              * 0 1 1 1 1 1 0 1
              -----------------
                1 1 0 1 0 1 0 1
              0 0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1 0
          1 1 0 1 0 1 0 1 0
        1 1 0 1 0 1 0 1 0
      1 1 0 1 0 1 0 1 0
    1 1 0 1 0 1 0 1 0
  0 0 0 0 0 0 0 0 0
-------------------------------
0 1 1 0 1 0 0 0 0 0 0 0 0 0 0 1
```



## Divide-and-Conquer Multiplication: Warmup

To multiply two n-digit integers:

- Multiply four ½n-digit integers.
- Add two ½n-digit integers, and shift to obtain result.

$$\begin{aligned} x &= 2^{n/2} \cdot x_1 + x_0 \\ y &= 2^{n/2} \cdot y_1 + y_0 \\ xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\ &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \end{aligned}$$

```
          1 1 0 1  0 1 0 1   y1 y0
        * 0 1 1 1  1 1 0 1   x1 x0
        ------------------
                   0 1 0 0  0 0 0 1   x0·y0
          1 0 1 0  1 0 0 1            x0·y1
          0 0 1 0  0 0 1 1            x1·y0
 0 1 0 1  1 0 1 1                     x1·y1
 -------------------------------------
 0 1 1 0  1 0 0 0  0 0 0 0  0 0 0 1
```

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)$$

↑ assumes n is a power of 2



## Key trick: 2 multiplies for the price of 1:

$$\begin{aligned} x &= 2^{n/2} \cdot x_1 + x_0 \\ y &= 2^{n/2} \cdot y_1 + y_0 \\ xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\ &= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \end{aligned}$$

Well, ok, 4 for 3 is more accurate...

$$\begin{aligned} \alpha &= x_1 + x_0 \\ \beta &= y_1 + y_0 \\ \alpha\beta &= \left(x_1 + x_0\right)\left(y_1 + y_0\right) \\ &= x_1 y_1 + \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\ \left(x_1 y_0 + x_0 y_1\right) &= \alpha\beta - x_1 y_1 - x_0 y_0 \end{aligned}$$

## Karatsuba Multiplication

To multiply two n-digit integers:

- Add two $\frac{1}{2}$n digit integers.
- Multiply three $\frac{1}{2}$n-digit integers.
- Add, subtract, and shift $\frac{1}{2}$n-digit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2}\cdot x_1 + x_0 \\
y &= 2^{n/2}\cdot y_1 + y_0 \\
xy &= 2^{n}\cdot x_1 y_1 + 2^{n/2}\cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
&= 2^{n}\cdot \underset{A}{x_1 y_1} + 2^{n/2}\cdot \big( \underset{B}{(x_1+x_0)(y_1+y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0} \big) + \underset{C}{x_0 y_0}
\end{aligned}
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$
T(n) \le \underbrace{T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(1+\lceil n/2 \rceil)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}
$$

$$
\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)
$$

$$
\Rightarrow T(n) = O(n^{\log_2 3}) = O(n^{1.585})
$$



## Multiplication – The Bottom Line

Naïve: $\Theta(n^2)$

Karatsuba: $\Theta(n^{1.59\ldots})$

Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems => $\Theta(n^{1.46\ldots})$

Best known: $\Theta$(n log n loglog n)

"Fast Fourier Transform"

but mostly unused in practice (unless you need really big numbers - a billion digits of $\pi$, say)

High precision arithmetic *IS* important for crypto



## Recurrences

Where they come from,
how to find them (above)

Next: how to solve them



## Mergesort (review)

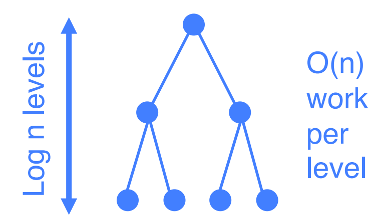
Mergesort: (recursively) sort 2 half-lists, then merge results.

$T(n) = 2T(n/2)+cn, \; n\geq 2$

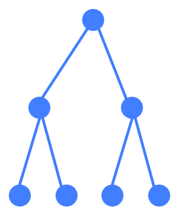
$T(1) = 0$

Solution: $\Theta$(n log n)

(details ~~later~~) now



## Solve: $T(1) = c$
## $T(n) = 2\,T(n/2) + cn$



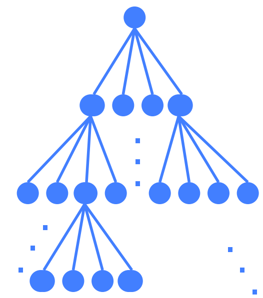
| Level | Num | Size | Work |
|---|---|---|---|
| 0 | $1 = 2^0$ | n | cn |
| 1 | $2 = 2^1$ | n/2 | 2cn/2 |
| 2 | $4 = 2^2$ | n/4 | 4cn/4 |
| … | … | … | … |
| i | $2^i$ | $n/2^i$ | $2^i\ c\ n/2^i$ |
| … | … | … | … |
| k-1 | $2^{k-1}$ | $n/2^{k-1}$ | $2^{k-1}\ c\ n/2^{k-1}$ |
| k | $2^k$ | $n/2^k = 1$ | $2^k\ T(1)$ |

$n = 2^k$ ; $k = \log_2 n$

Total Work: $c\ n \log_2 n$ (add last col)



## Solve: $T(1) = c$
## $T(n) = 4\,T(n/2) + cn$



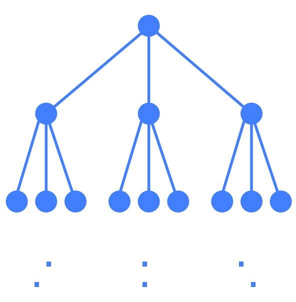
| Level | Num | Size | Work |
|---|---|---|---|
| 0 | $1 = 4^0$ | n | cn |
| 1 | $4 = 4^1$ | n/2 | 4cn/2 |
| 2 | $16 = 4^2$ | n/4 | 16cn/4 |
| … | … | … | … |
| i | $4^i$ | $n/2^i$ | $4^i\ c\ n/2^i$ |
| … | … | … | … |
| k-1 | $4^{k-1}$ | $n/2^{k-1}$ | $4^{k-1}\ c\ n/2^{k-1}$ |
| k | $4^k$ | $n/2^k = 1$ | $4^k\ T(1)$ |

$n = 2^k$ ; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 4^i cn/2^i = O(n^2)$



## Solve: $T(1) = c$
## $T(n) = 3\,T(n/2) + cn$



| Level | Num | Size | Work |
|---|---|---|---|
| 0 | $1 = 3^0$ | n | cn |
| 1 | $3 = 3^1$ | n/2 | 3cn/2 |
| 2 | $9 = 3^2$ | n/4 | 9cn/4 |
| … | … | … | … |
| i | $3^i$ | $n/2^i$ | $3^i\ c\ n/2^i$ |
| … | … | … | … |
| k-1 | $3^{k-1}$ | $n/2^{k-1}$ | $3^{k-1}\ c\ n/2^{k-1}$ |
| k | $3^k$ | $n/2^k = 1$ | $3^k\ T(1)$ |

$n = 2^k$ ; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 3^i cn/2^i$



## Solve: $T(1) = c$
## $T(n) = 3\,T(n/2) + cn$ (cont.)

$$T(n) = \sum_{i=0}^{k} 3^i cn/2^i$$
$$= cn \sum_{i=0}^{k} 3^i/2^i$$
$$= cn \sum_{i=0}^{k} \left(\tfrac{3}{2}\right)^i$$
$$= cn \frac{\left(\tfrac{3}{2}\right)^{k+1} - 1}{\left(\tfrac{3}{2}\right) - 1}$$

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1} \quad (x \neq 1)$$

## Solve: $T(1) = c$
## $T(n) = 3\,T(n/2) + cn$ (cont.)

$$= 2cn\left(\left(\tfrac{3}{2}\right)^{k+1} - 1\right)$$

$$< 2cn\left(\tfrac{3}{2}\right)^{k+1}$$

$$= 3cn\left(\tfrac{3}{2}\right)^{k}$$

$$= 3cn\,\frac{3^k}{2^k}$$



## Solve: $T(1) = c$
## $T(n) = 3\,T(n/2) + cn$ (cont.)

$$= 3cn\,\frac{3^{\log_2 n}}{2^{\log_2 n}}$$

$$= 3cn\,\frac{3^{\log_2 n}}{n}$$

$$= 3c\,3^{\log_2 n}$$

$$= 3c\left(n^{\log_2 3}\right)$$

$$= O\left(n^{1.59\ldots}\right)$$

$$a^{\log_b n}$$
$$= \left(b^{\log_b a}\right)^{\log_b n}$$
$$= \left(b^{\log_b n}\right)^{\log_b a}$$
$$= n^{\log_b a}$$



## Divide and Conquer
## Master Recurrence

If $T(n) = aT(n/b)+cn^k$ for $n > b$ then

- if $a > b^k$ then $T(n)$ is $\Theta(n^{\log_b a})$ — [many subproblems => leaves dominate]
- if $a < b^k$ then $T(n)$ is $\Theta(n^k)$ — [few subproblems => top level dominates]
- if $a = b^k$ then $T(n)$ is $\Theta(n^k \log n)$ — [balanced => all log n levels contribute]

True even if it is $\lceil n/b \rceil$ instead of $n/b$.



## D & C Summary

Idea:

- "Two halves are better than a whole"
  - if the base algorithm has super-linear complexity.
- "If a little's good, then more's better"
  - repeat above, recursively

Analysis: recursion tree or Master Recurrence

Applications: Many.

- Binary Search, Merge Sort, (Quicksort), Closest points, Integer multiply,…