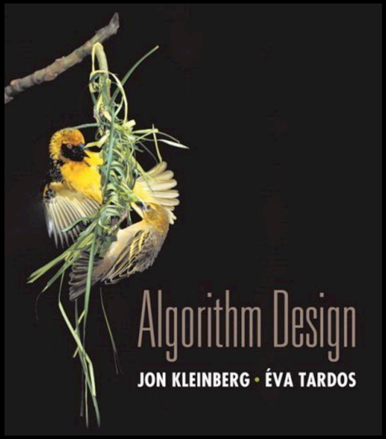


Chapter 4

Greedy Algorithms

Slides by Kevin Wayne.

Intro: Coin Changing

Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

Ex: 34¢.

Algorithm is "Greedy": One large coin better than two or more smaller ones

Cashier's algorithm. At each iteration, give the *largest* coin valued ≤ the amount to be paid.

Ex: $2.89.

Coin-Changing: Does Greedy Always Work?

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

- Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
- Optimal: 70, 70.

Algorithm is "Greedy", but also short-sighted – attractive choice now may lead to dead ends later.

Correctness is key!

Outline & Goals

"Greedy Algorithms"
- what they are

Pros
- intuitive
- often simple
- often fast

Cons
- often incorrect!

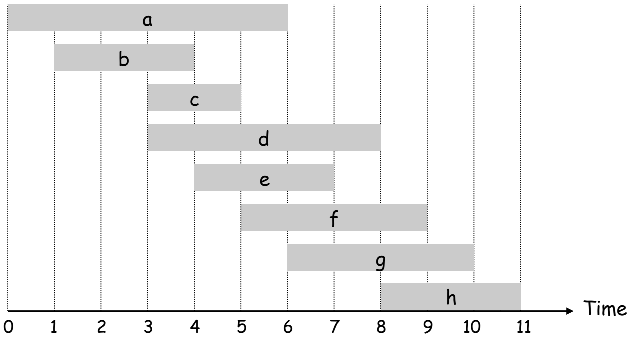
Proof techniques
- stay ahead
- structural
- exchange arguments

4.1 Interval Scheduling

Proof Technique 1: "greedy stays ahead"

Interval Scheduling

Interval scheduling.
- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- What order? Does that give best answer? Why or why not? Does it help to be greedy about order?

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of start time s_j.
- [Earliest finish time] Consider jobs in ascending order of finish time f_j.
- [Shortest interval] Consider jobs in ascending order of interval length $f_j - s_j$.
- [Fewest conflicts] For each job, count the number of conflicting jobs c_j. Schedule in ascending order of conflicts c_j.

Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some order. Take each job provided it's compatible with the ones already taken.

Interval Scheduling: Greedy Algorithm

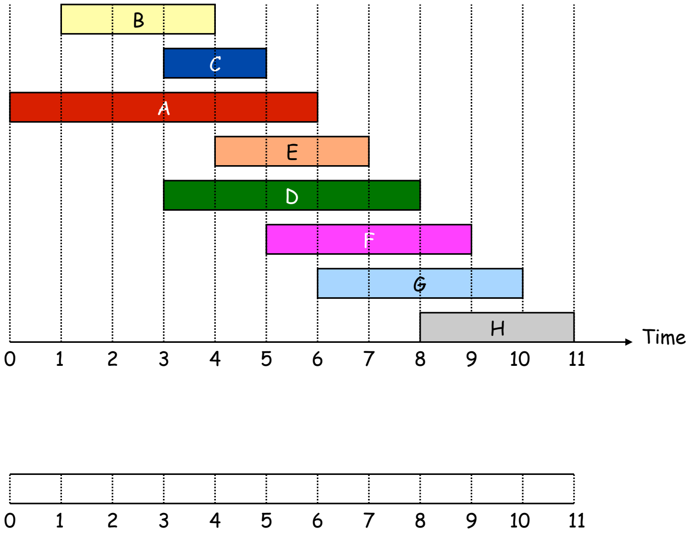
Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that $f_1 \le f_2 \le \ldots \le f_n$.

jobs selected
A ← $\phi$
for j = 1 to n {
   if (job j compatible with A)
      A ← A $\cup$ {j}
}
return A
```

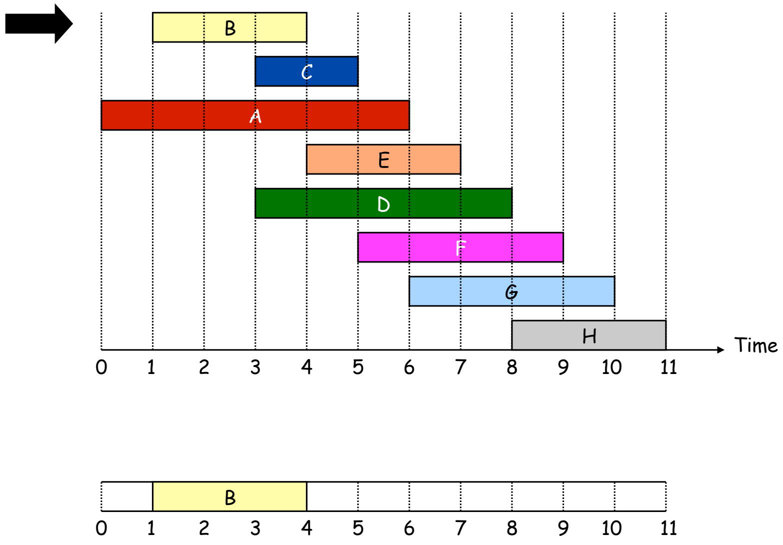
Implementation. O(n log n).

- Remember job j* that was added last to A.
- Job j is compatible with A if $s_j \ge f_{j^*}$.

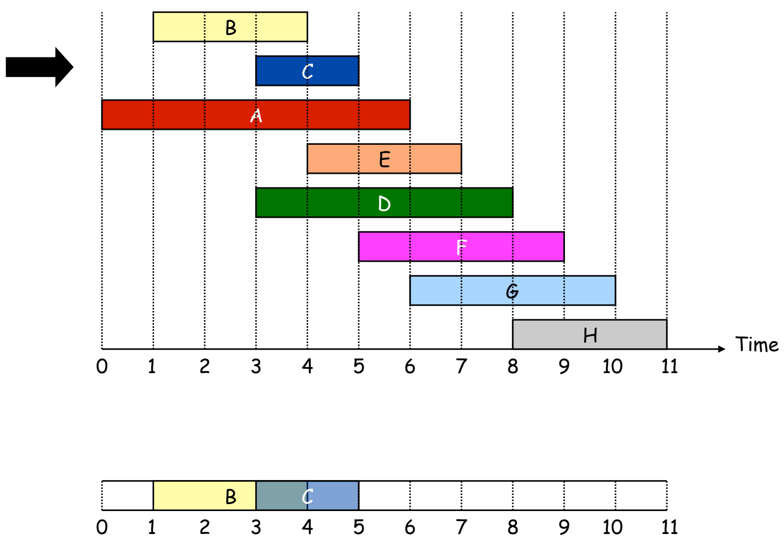
Interval Scheduling



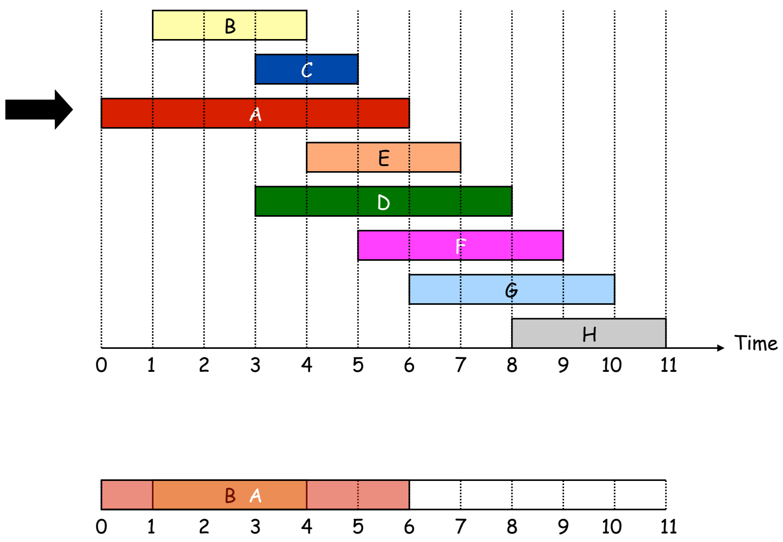
Interval Scheduling



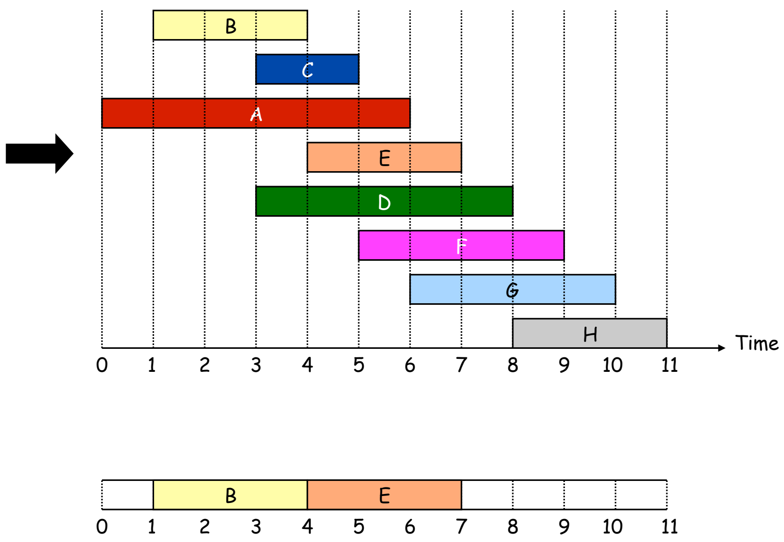
Interval Scheduling



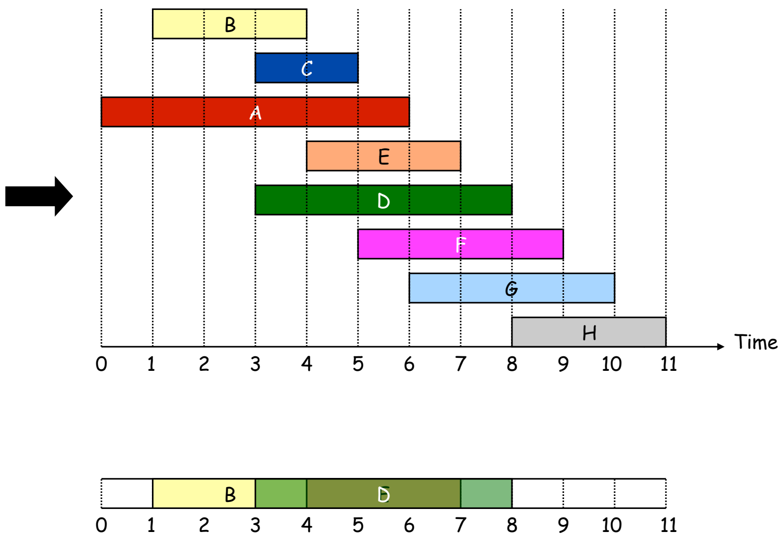
Interval Scheduling



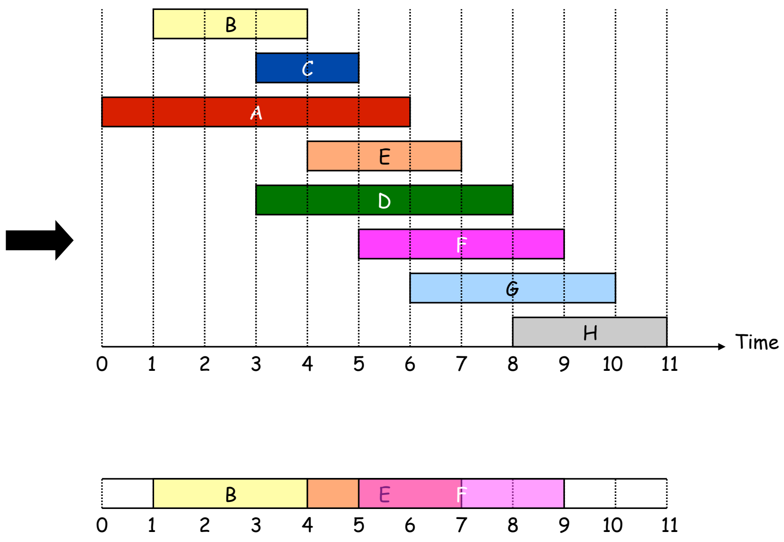
Interval Scheduling



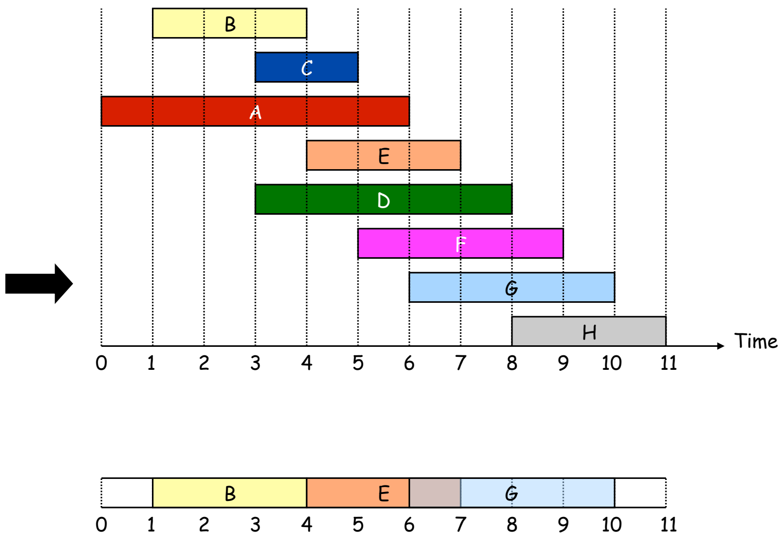
Interval Scheduling



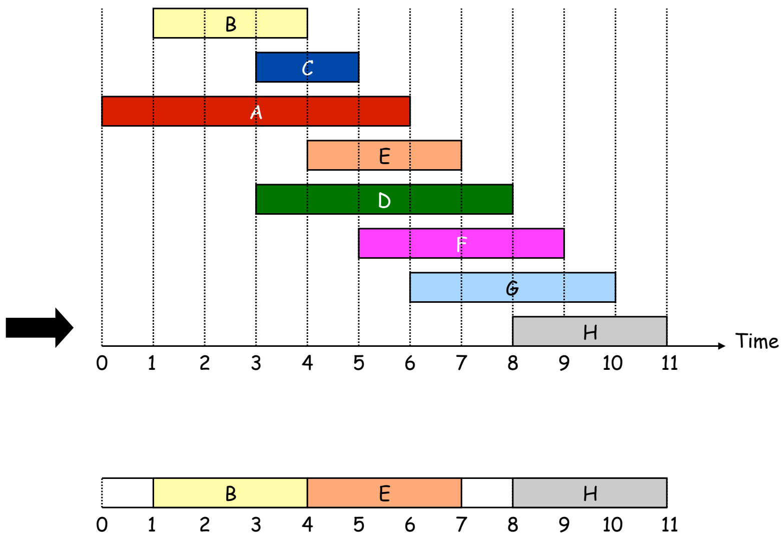
Interval Scheduling



Interval Scheduling



Interval Scheduling



Interval Scheduling: Correctness

Theorem. Greedy algorithm is optimal.

Pf. ("greedy stays ahead")

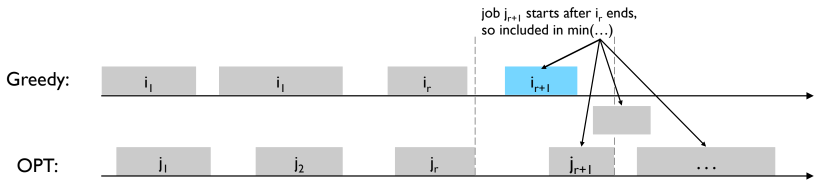
Let $i_1, i_2, \ldots i_k$ be jobs picked by greedy, $j_1, j_2, \ldots j_m$ those in some optimal solution

Show $f(i_r) \le f(j_r)$ by induction on r.

Basis: i_1 chosen to have min finish time, so $f(i_1) \le f(j_1)$

Ind: $f(i_r) \le f(j_r) \le s(j_{r+1})$, so j_{r+1} is among the candidates considered by greedy when it picked i_{r+1}, & it picks min finish, so $f(i_{r+1}) \le f(j_{r+1})$

Similarly, $k \ge m$, else j_{k+1} is among (nonempty) set of candidates for i_{k+1}


job j_{r+1} starts after i_r ends, so included in min(…)

4.1 Interval Partitioning

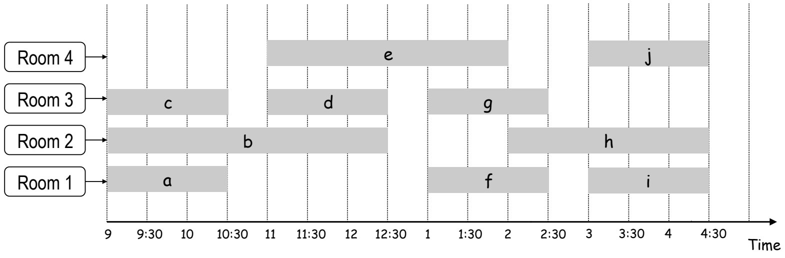
Proof Technique 2: "Structural"

Interval Partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

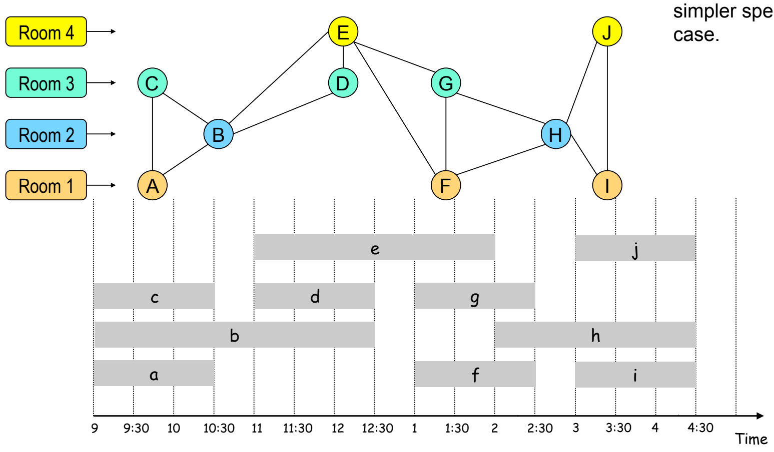
Ex: This schedule uses 4 classrooms to schedule 10 lectures.



Interval Partitioning as Interval Graph Coloring

Vertices = classes;
edges = conflicting class pairs;
different colors = different assigned rooms

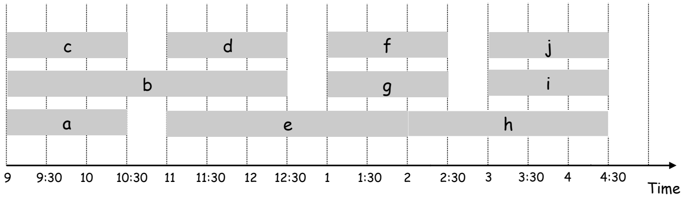
Note: graph coloring is very hard in general, but graphs corresponding to interval intersections are a much simpler special case.



Interval Partitioning

Interval partitioning.
- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses only 3.



Interval Partitioning: A "Structural" Lower Bound on Optimal Solution

Def. The depth of a set of open intervals is the maximum number that contain any given time.

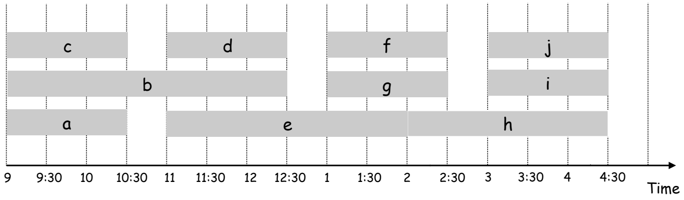
no collisions at ends

Key observation. Number of classrooms needed ≥ depth.

Ex: Depth of schedule below = 3 ⇒ schedule below is optimal.

a, b, c all contain 9:30

Q. Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that $s_1 \le s_2 \le ... \le s_n$.
d ← 0    ← number of allocated classrooms

for j = 1 to n {
   if (lect j is compatible with some classroom k, 1≤k≤d)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation? Run-time?
Next HW

Interval Partitioning: Greedy Analysis

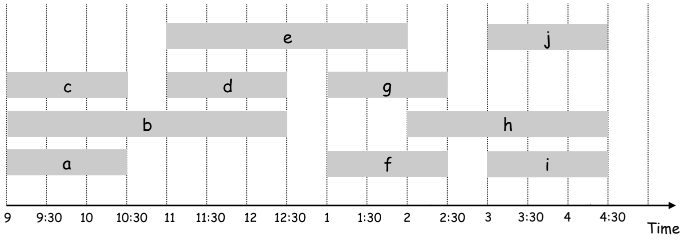
Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.
Pf (exploit structural property).
- Let d = number of classrooms that the greedy algorithm allocates.
- Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 previously used classrooms.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$, i.e. depth ≥ d
- "Key observation" ⇒ all schedules use ≥ depth classrooms, so d = depth and greedy is optimal ▪

Interval Partitioning: Alt Proof (An "Exchange Argument")

- When 4th room added, room 1 was free; why not swap it in there?
- (A: it conflicts with later stuff in schedule, which dominoes)
- But: room 4 schedule after 11:00 is conflict-free; so is room 1 schedule, so could swap both post-11:00 schedules
- Why does it help? Delays needing 4th room; repeat.

Cleaner: "Let S* be an opt sched with latest use of last room. When that room is added, all others in use (else we could swap, contradicting 'latest') so #rooms = depth, hence optimal"



4.2 Scheduling to Minimize Lateness

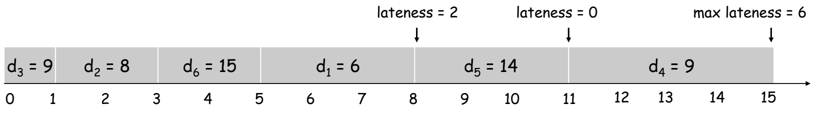
Scheduling to Minimize Lateness

Minimizing lateness problem.

- Single resource processes one job at a time.
- Job j requires t_j units of processing time and is due at time d_j.
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$.
- Goal: schedule all jobs to minimize maximum lateness $L = \max \ell_j$.

Ex:

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first]
Consider jobs in ascending order of processing time t_j.

[Earliest deadline first]
Consider jobs in ascending order of deadline d_j.

[Smallest slack]
Consider jobs in ascending order of slack $d_j - t_j$.

Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

[Shortest processing time first] Consider jobs in ascending order of processing time t_j.

	1	2
t_j	1	10
d_j	100	10

counterexample

[Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	2	10

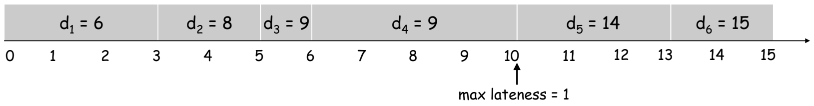
counterexample

Minimizing Lateness: Greedy Algorithm

Greedy algorithm. Earliest deadline first.

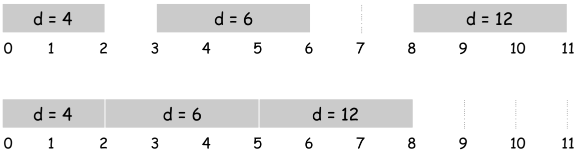
```
Sort n jobs by deadline so that $d_1 \le d_2 \le … \le d_n$

$t \leftarrow 0$
for j = 1 to n
   // Assign job j to interval [t, t + $t_j$]:
   $s_j \leftarrow t$, $f_j \leftarrow t + t_j$
   $t \leftarrow t + t_j$
output intervals [$s_j$, $f_j$]
```

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: No Idle Time

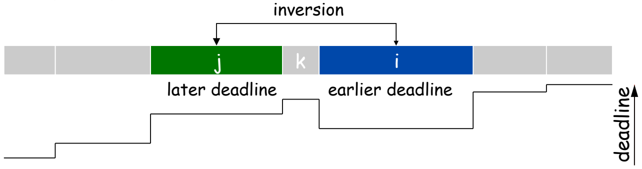
Observation. There exists an optimal schedule with no idle time.



Observation. The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Def. An *inversion* in schedule S is a pair of jobs i and j such that: deadline i < j but j scheduled before i.



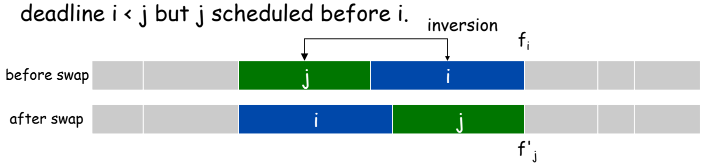
Observation. Greedy schedule has no inversions.

Observation. If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.
(If j & i aren't consecutive, then look at the job k scheduled right after j. If $d_k < d_j$, then (j,k) is a consecutive inversion; if not, then (k,i) is an inversion, & nearer to each other - repeat.)

Observation. Swapping *adjacent* inversion reduces # inversions by 1 (exactly)

Minimizing Lateness: Inversions

Def. An inversion in schedule S is a pair of jobs i and j such that: deadline i < j but j scheduled before i.



(j had later deadline, so is less tardy than i was)

Claim. *Swapping* two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards.

- $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \le \ell_i$
- If job j is now late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)}\\ &= f_i - d_j && (j \text{ finishes at time } f_i)\\ &\le f_i - d_i && (d_i \le d_j)\\ &= \ell_i && \text{(definition)}\end{aligned}$$

only j moves later, but it's no later than i was, so max not increased

---

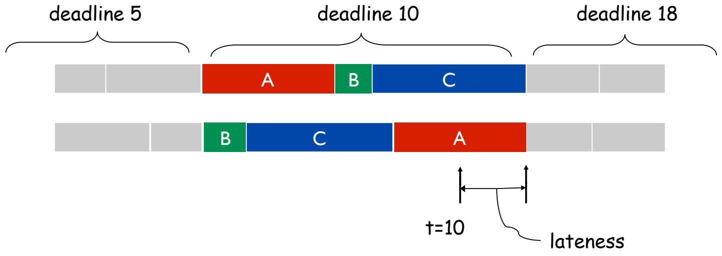
Minimizing Lateness: No Inversions

Claim. All inversion-free schedules S have the same max lateness

Pf. If S has no inversions, then deadlines of scheduled jobs are monotonically nondecreasing, i.e., they increase (or stay the same) as we walk through the schedule from left to right.

Two such schedules can differ only in the order of jobs with the same deadlines.

Within a group of jobs with the same deadline, the max lateness is the lateness of the last job in the group - order within the group doesn't matter.



Minimizing Lateness: Correctness of Greedy Algorithm

Theorem. Greedy schedule S is optimal

Pf. Let S* be an optimal schedule with the fewest number of inversions

Can assume S* has no idle time.

If S* has an inversion, let i-j be an adjacent inversion

Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions

This contradicts definition of S*

So, S* has no inversions. But then Lateness(S) = Lateness(S*)

Greedy Analysis Strategies

Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

4.3 Optimal Caching

^{1}**cache**

Pronunciation: 'kash
Function: *noun*
Etymology: French, from *cacher* to press, hide

a hiding place especially for concealing and preserving provisions or implements

^{2}**cache**

Function: *transitive verb*

to place, hide, or store in a cache

-Webster's Dictionary

Optimal Offline Caching

Caching.
- Cache with capacity to store k items.
- Sequence of m item requests $d_1, d_2, \ldots, d_m$.
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

Goal. Eviction schedule that minimizes number of cache misses.

Ex: k = 2, initial cache = ab,
requests: a, b, c, b, c, a, a, b.
Optimal eviction schedule: 2 cache misses.

requests	cache	
a	a	b
b	a	b
c	🟥 c	b
b	c	b
c	c	b
a	🟥 a	b
a	a	b
b	a	b

Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache: a b c d e f

future queries: g a b c e d a b b a c d e a f a d e f g h ...

↑ cache miss (g) ↑ eject this one (f)

Theorem. [Bellady, 1960s] FF is optimal eviction schedule.
Pf. Algorithm and theorem are intuitive; proof is subtle.

Motivation: "Online" problem is typically what's needed in practice - decide what to evict *without* seeing the future. How to evaluate such an alg? Fewer misses is obviously better, but how few? FF is a useful benchmark - best online alg is unknown, but it's no better than FF, so online performance close to FF's is the best you can hope for.

4.4 Shortest Paths in a Graph

You've seen this in 326 or 373, so this section and next two on min spanning tree are review. I won't lecture on them, but you should review the material. Both, but especially shortest paths, are common problems, having many applications.

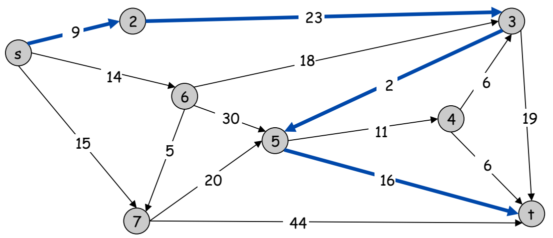
Shortest Path Problem

Shortest path network.

- Directed graph G = (V, E).
- Source s, destination t.
- Length ℓ_e = length of edge e.

Shortest path problem: find shortest directed path from s to t.

cost of path = sum of edge costs in path



Cost of path s-2-3-5-t
= 9 + 23 + 2 + 16
= 48.

Dijkstra's Algorithm

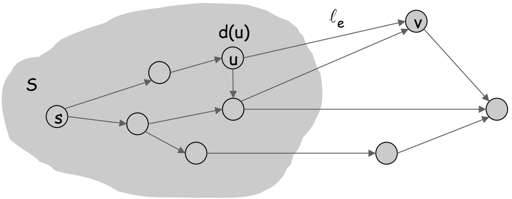
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e = (u,v)\,:\,u \in S} d(u) + \ell_e,$$

add v to S, and set d(v) = π(v).

shortest path to some u in explored part, followed by a single edge (u, v)



Dijkstra's Algorithm

Dijkstra's algorithm.

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