CSE 417: Algorithms and Computational Complexity

Winter 2009

Graphs and Graph Algorithms

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Goals

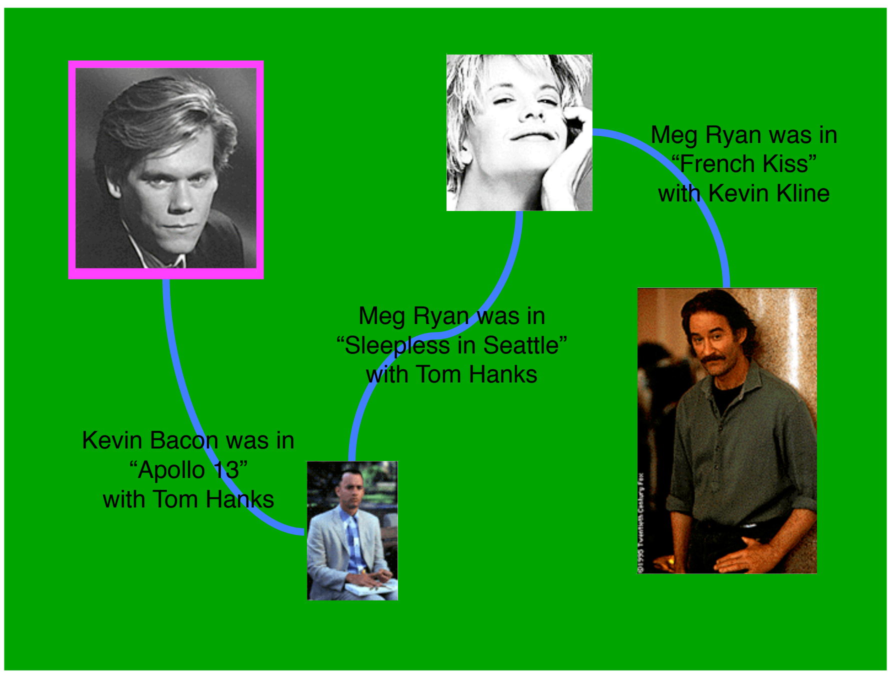
Graphs: defns, examples, utility, terminology

Representation: input, internal

Traversal: Breadth- & Depth-first search

Three Algorithms:

- Connected components
- Bipartiteness
- Topological sort



Objects & Relationships

The Kevin Bacon Game:

- Actors
- Two are related if they've been in a movie together

Exam Scheduling:

- Classes
- Two are related if they have students in common

Traveling Salesperson Problem:

- Cities
- Two are related if can travel *directly* between them

Graphs

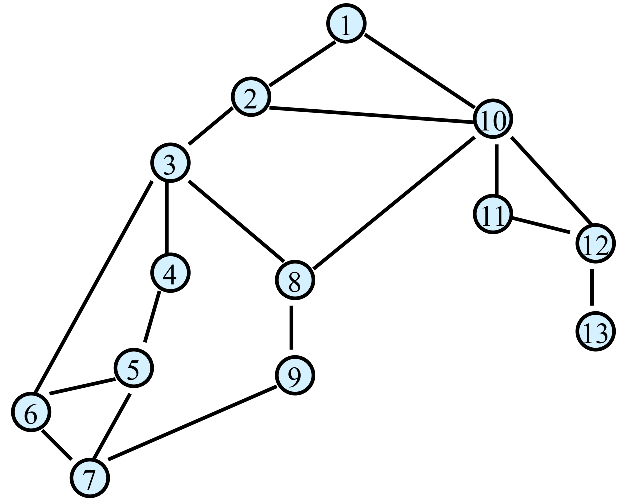
An extremely important formalism for representing (binary) relationships

Objects: “vertices”, aka “nodes”

Relationships between pairs: “edges”, aka “arcs”

Formally, a graph G = (V, E) is a pair of sets, V the vertices and E the edges

Undirected Graph G = (V,E)



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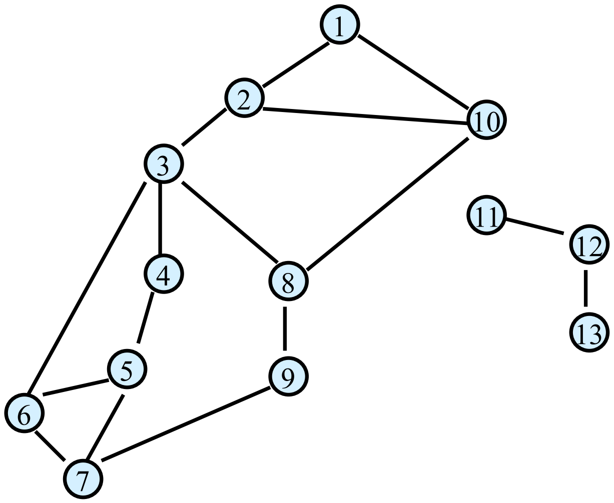

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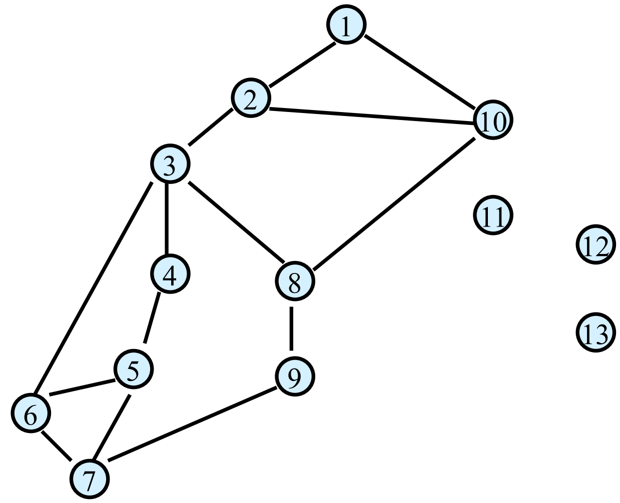

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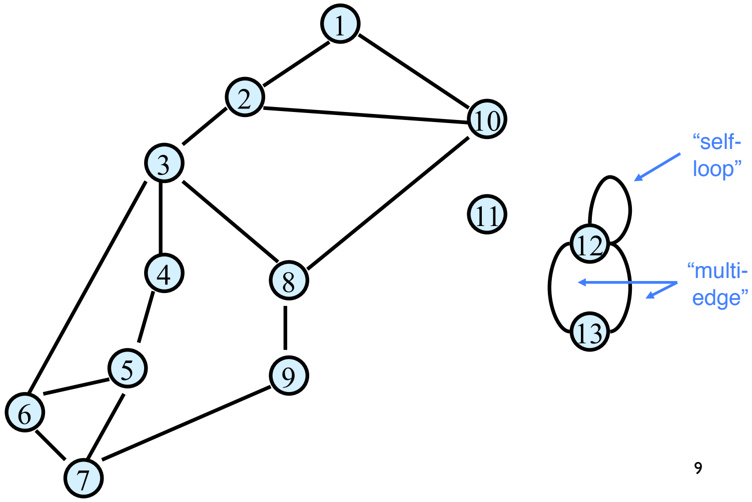

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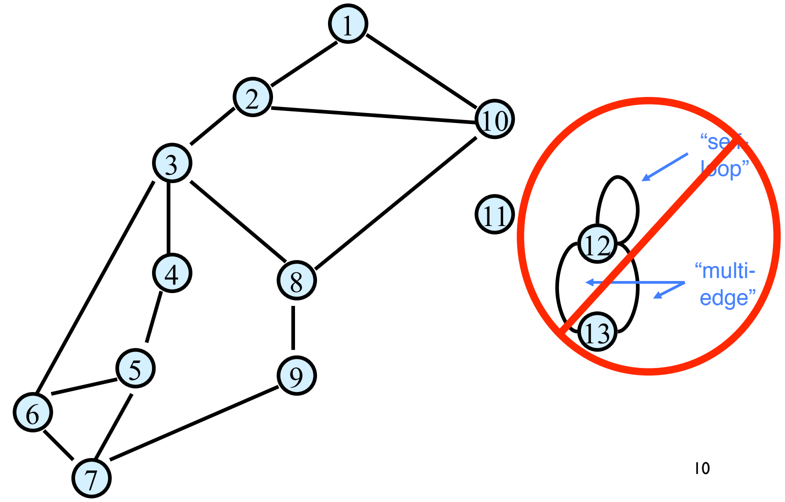
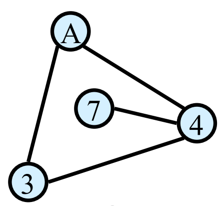
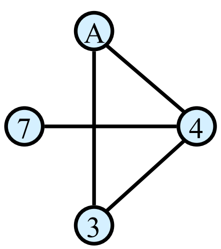
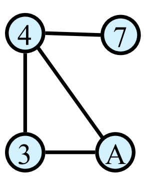


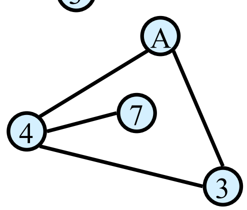
Graphs don’t live in Flatland

Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.

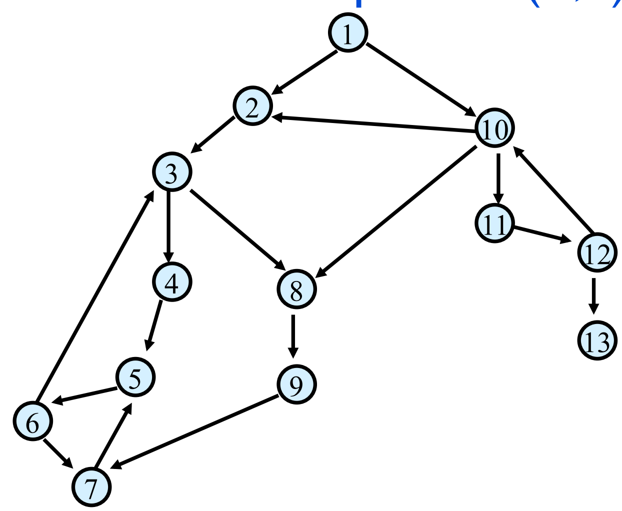




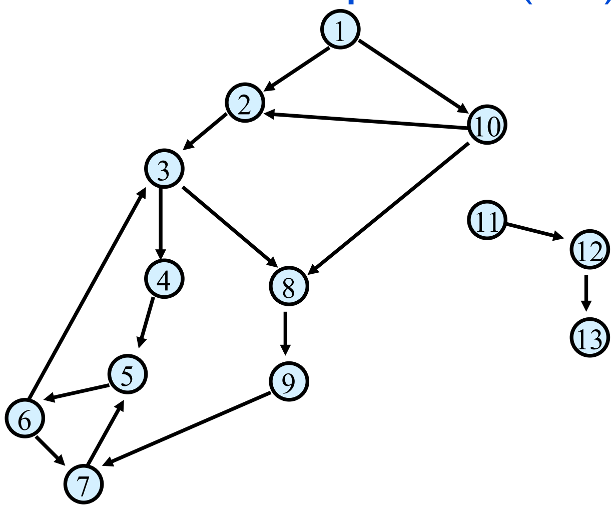




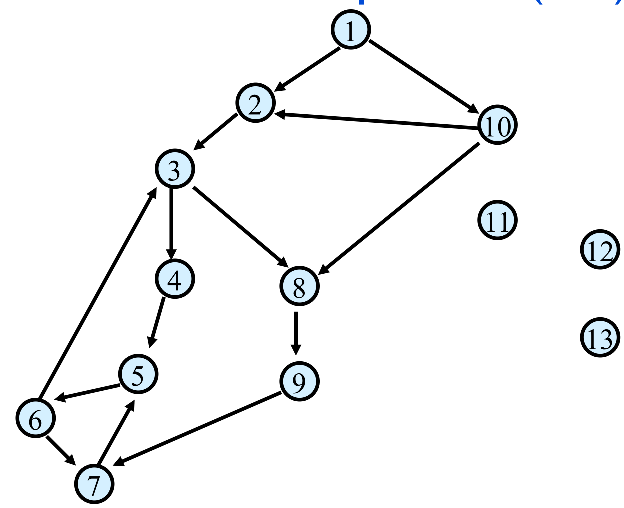
Directed Graph G = (V,E)



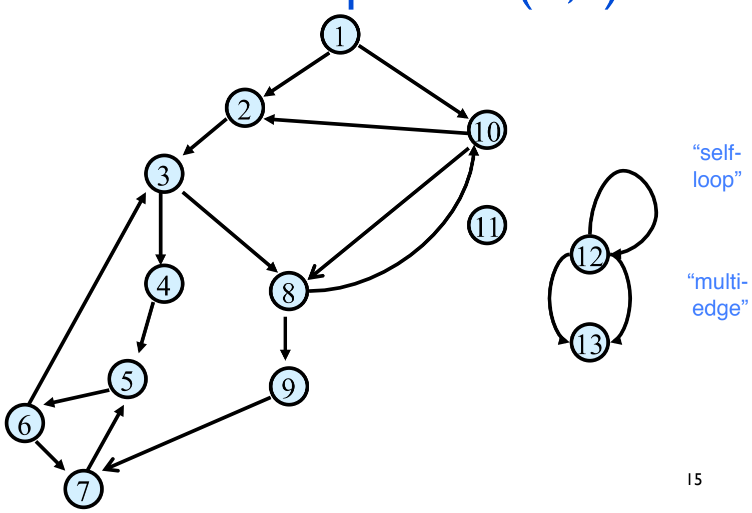
Directed Graph G = (V,E)



Directed Graph G = (V,E)



Directed Graph G = (V,E)



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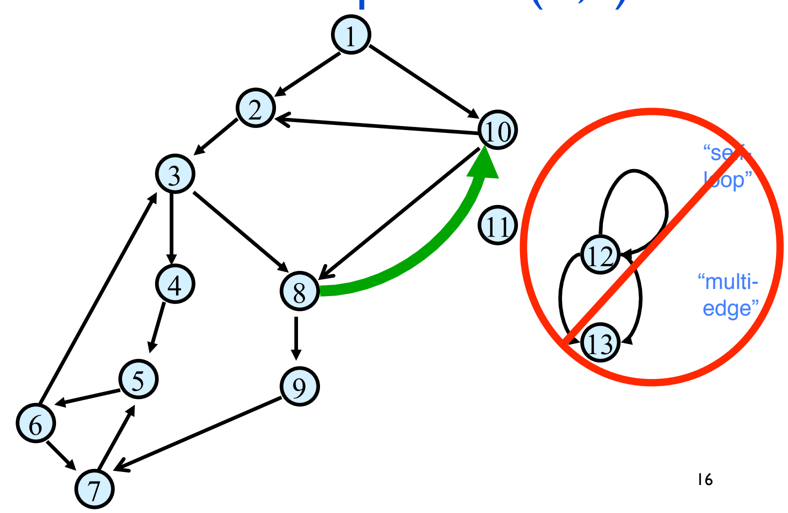
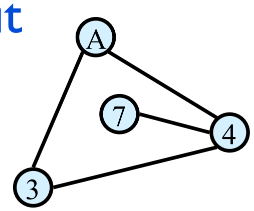


Specifying undirected graphs as input



What are the vertices?

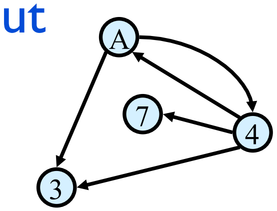
- Explicitly list them: {"A", "7", "3", "4"}

What are the edges?

- Either, set of edges {{A,3}, {7,4}, {4,3}, {4,A}}
- Or, (symmetric) adjacency matrix:

	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

Specifying directed graphs as input



What are the vertices?

- Explicitly list them: {"A", "7", "3", "4"}

What are the edges?

- Either, set of directed edges: {(A,4), (4,7), (4,3), (4,A), (A,3)}
- Or, (nonsymmetric) adjacency matrix:

	A	7	3	4
A	0	0	1	1
7	0	0	0	0
3	0	0	0	0
4	1	1	1	0

Vertices vs # Edges

Let G be an undirected graph with n vertices and m edges. How are n and m related?

Since

- every edge connects two different vertices (no loops), and no two edges connect the same two vertices (no multi-edges),

it must be true that:

$$\boxed{0 \leq m \leq n(n-1)/2 = O(n^2)}$$

More Cool Graph Lingo

A graph is called *sparse* if $m \ll n^2$, otherwise it is *dense*

- Boundary is somewhat fuzzy; $O(n)$ edges is certainly sparse, $\Omega(n^2)$ edges is dense.

Sparse graphs are common in practice

- E.g., all planar graphs are sparse ($m \leq 3n-6$, for $n \geq 3$)

Q: which is a better run time, $O(n+m)$ or $O(n^2)$?

A: $O(n+m) = O(n^2)$, but n+m usually way better!

Representing Graph G = (V,E)

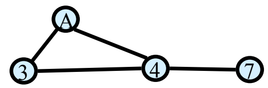
internally, indp of input format

Vertex set $V = \{v_1, \ldots, v_n\}$

Adjacency Matrix A

$A[i,j] = 1$ iff $(v_i,v_j) \in E$

Space is n^2 bits



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

Advantages:

O(1) test for presence or absence of edges.

Disadvantages: inefficient for sparse graphs, both in storage and access

$m \ll n^2$

Representing Graph G=(V,E)

n vertices, m edges

Adjacency List:

O(n+m) words

Advantages:

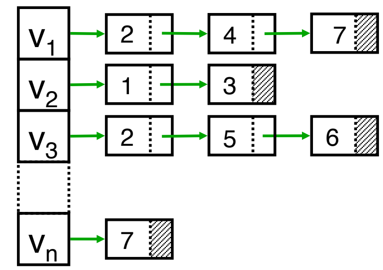
Compact for sparse graphs

Easily see all edges

Disadvantages

More complex data structure

no O(1) edge test

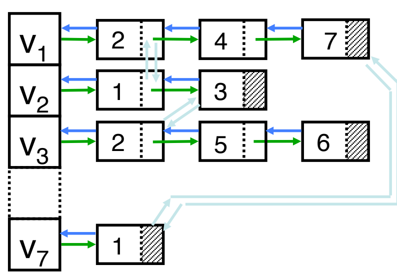


Representing Graph G=(V,E)

n vertices, m edges

Adjacency List:

O(n+m) words



Back- and cross pointers more work to build, but allow easier traversal and deletion of edges, *if needed,* (don't bother if not)

Graph Traversal

Learn the basic structure of a graph

"Walk," *via edges*, from a fixed starting vertex s to all vertices reachable from s

Being *orderly* helps. Two common ways:

Breadth-First Search

Depth-First Search

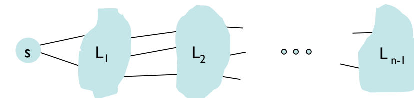
Breadth-First Search

Completely explore the vertices in order of their distance from *s*

Naturally implemented using a queue

Breadth-First Search

Idea: Explore from s in all possible directions, layer by layer.



BFS algorithm.

- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes not in L_0 or L_1, and having an edge to a node in L_1.
- L_{i+1} = all nodes not in earlier layers, and having an edge to a node in L_i.

Theorem. For each i, L_i consists of all nodes at distance (i.e., min path length) exactly i from s.

Cor: There is a path from s to t iff t appears in some layer.

Graph Traversal: Implementation

Learn the basic structure of a graph

"Walk," via edges, from a fixed starting vertex *s* to all vertices reachable from *s*

Three states of vertices

- *undiscovered*
- *discovered*
- *fully-explored*

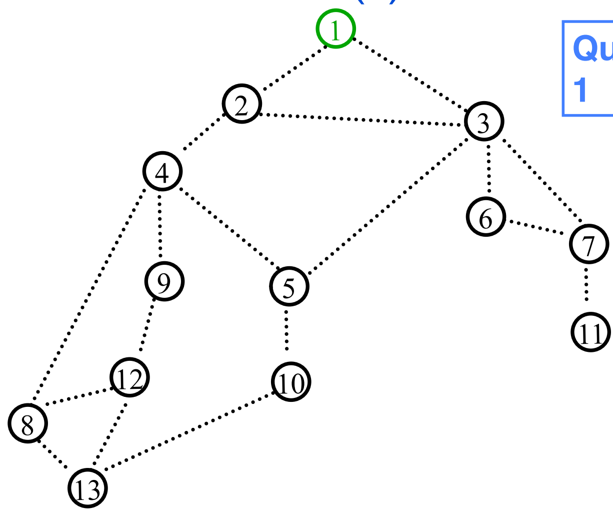
BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

```
BFS(s)
    mark  s "discovered"
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully explored
```

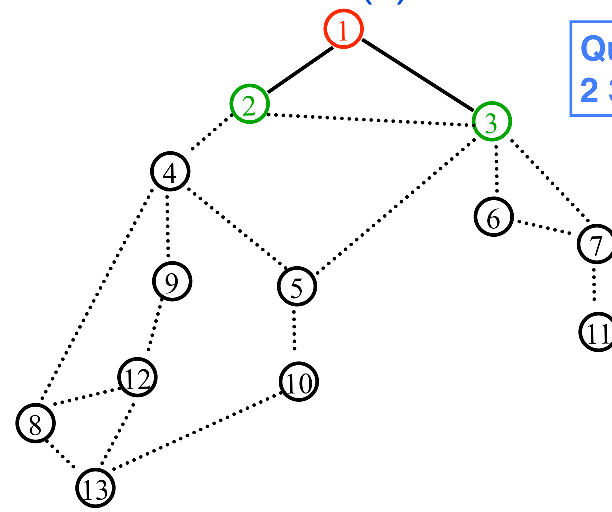
Exercise: modify code to number vertices & compute level numbers

BFS(v)



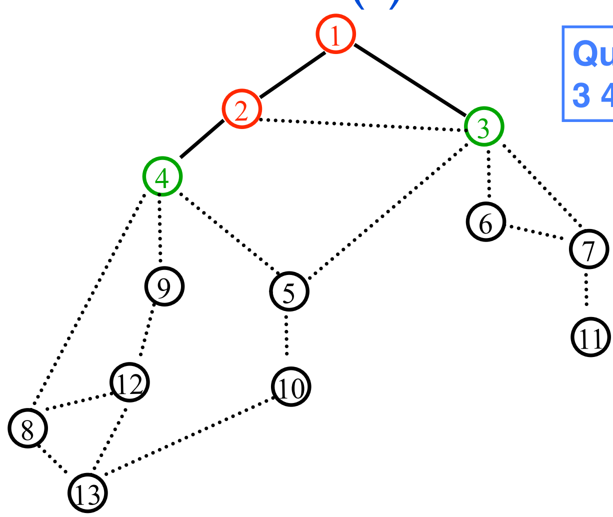
Queue:
1

BFS(v)



Queue:
2 3

BFS(v)



Queue:
3 4

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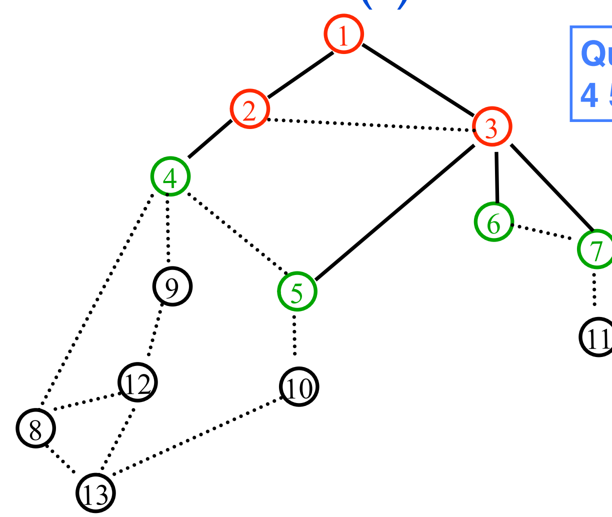
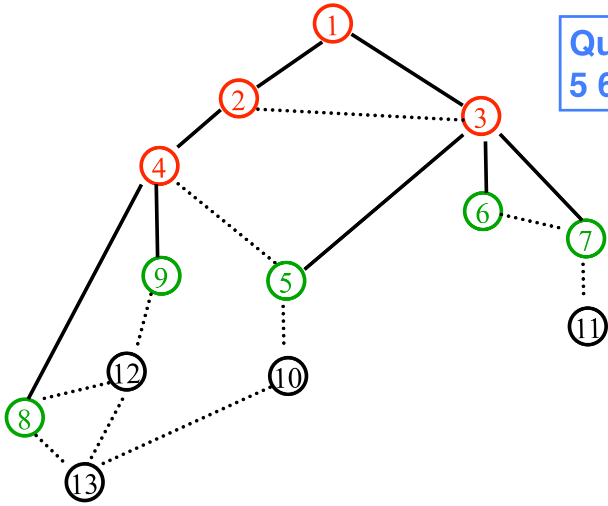


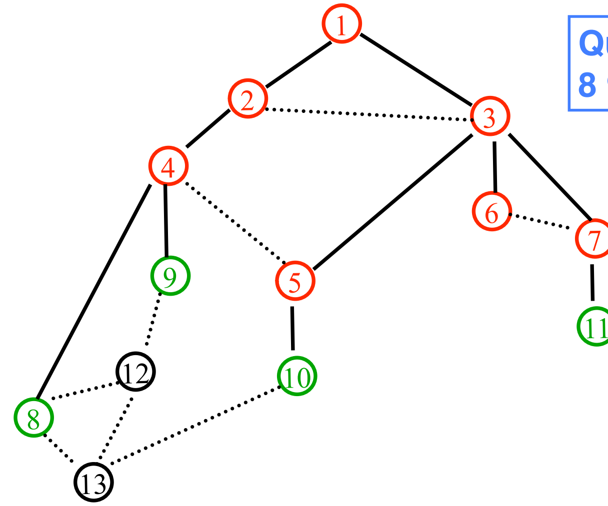
Queue:
4 5 6 7

BFS(v)



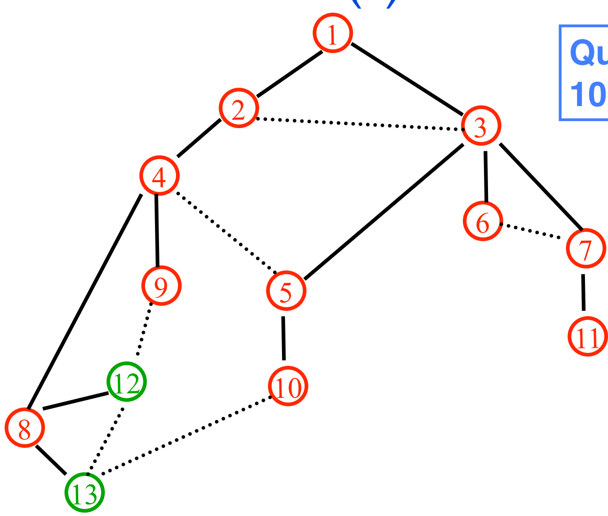
Queue:
5 6 7 8 9

BFS(v)



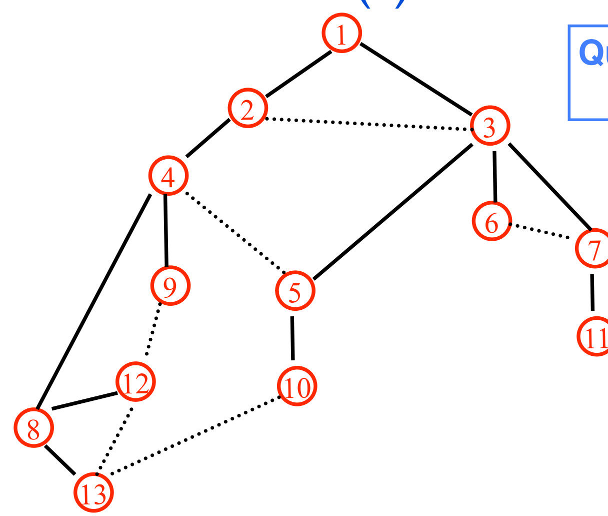
Queue:
8 9 10 11

BFS(v)



Queue:
10 11 12 13

BFS(v)



Queue:

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Exercise: modify code to number vertices & compute level numbers

BFS analysis

Each vertex is added to/removed from queue at most once

Each edge is explored once from each end-point

Total cost O(m), m = # of edges

Exercise: extend algorithm and analysis to non-connected graphs

Properties of (Undirected) BFS(v)

BFS(v) visits x if and only if there is a path in G from v to x.

Edges into then-undiscovered vertices define a ***tree*** – the "breadth first spanning tree" of G

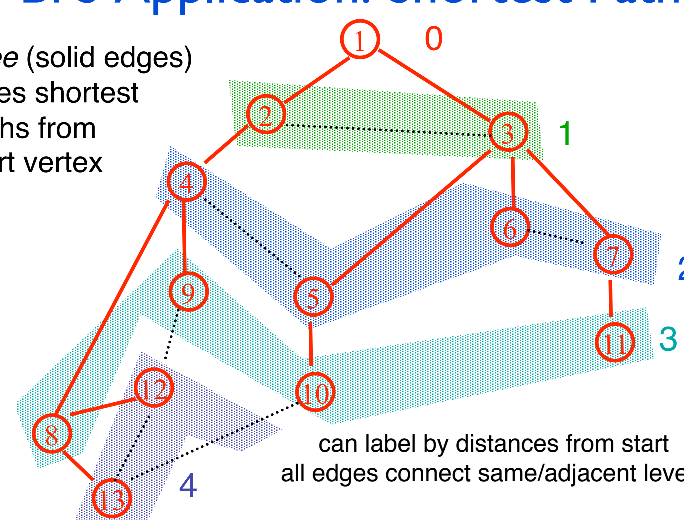
Level i in this tree are exactly those vertices *u* such that the shortest path (in G, not just the tree) from the root v is of length i.

All non-tree edges join vertices on the same or adjacent levels

not true of every spanning tree!

BFS Application: Shortest Paths

Tree (solid edges) gives shortest paths from start vertex



can label by distances from start
all edges connect same/adjacent levels

BFS Application: Shortest Paths

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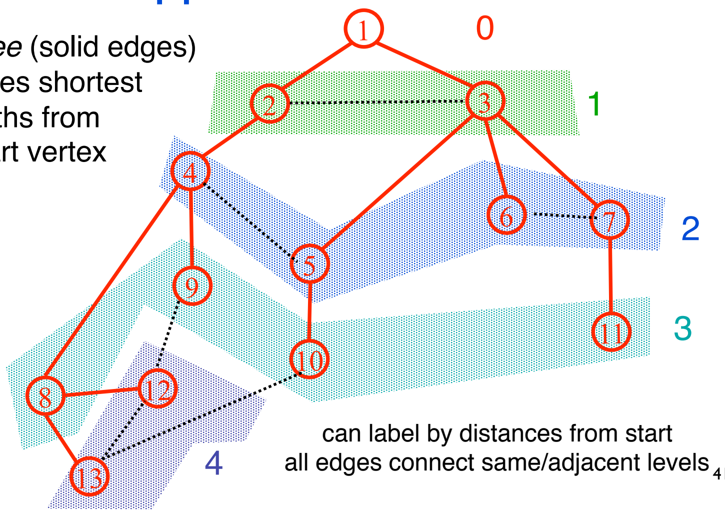

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BFS Application: Shortest Paths

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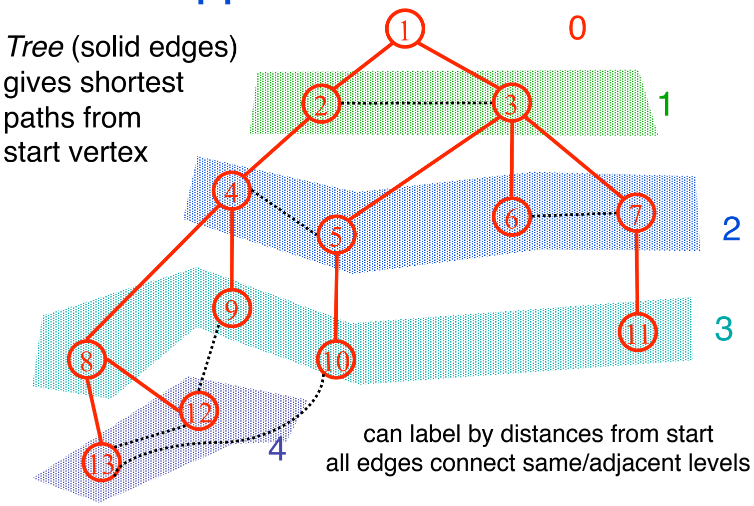

can label by distances from start
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BFS Application: Shortest Paths

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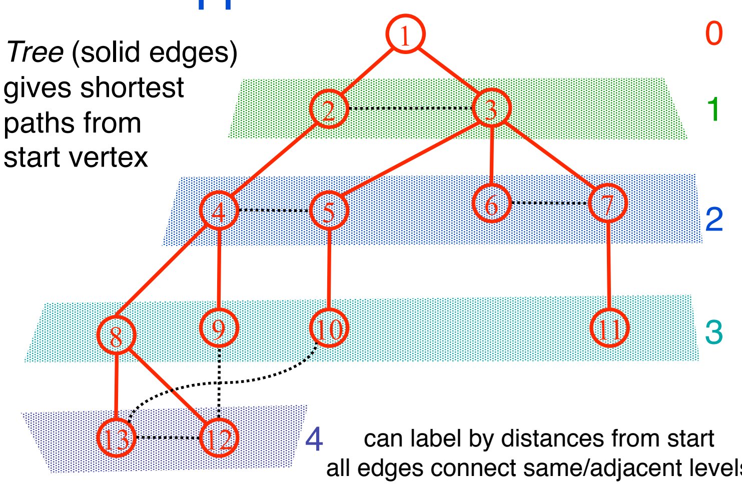


can label by distances from start
all edges connect same/adjacent levels

Why fuss about trees?

Trees are simpler than graphs

Ditto for algorithms on trees vs algs on graphs

So, this is often a good way to approach a graph problem: find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure

E.g., BFS finds a tree s.t. level-jumps are minimized

DFS (next) finds a different tree, but it also has interesting structure…

Graph Search Application: Connected Components

Want to answer questions of the form:

given vertices u and v, is there a path from u to v?

Idea: create array A such that

A[u] = smallest numbered vertex that is connected to u. Question reduces to whether A[u]=A[v]?

Q: Why not create 2-d array Path[u,v]?

Graph Search Application: Connected Components

initial state: all v undiscovered
for v = 1 to n do
  if state(v) != fully-explored then
    BFS(v): setting A[u] ←v for each u found
    (and marking u discovered/fully-explored)
  endif
endfor

Total cost: O(n+m)

each edge is touched a constant number of times (twice)

works also with DFS

3.4 Testing Bipartiteness

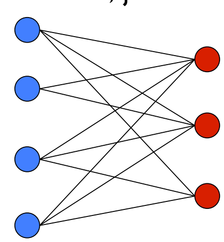
Bipartite Graphs

Def. An undirected graph G = (V, E) is *bipartite* if the nodes can be colored red or blue such that every edge has one red and one blue end.

"bi-partite" means "two parts." An equivalent definition: G is bipartitite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.

Applications.

Stable marriage: men = red, women = blue
Scheduling: machines = red, jobs = blue

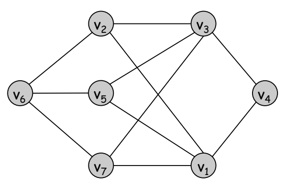


a bipartite graph

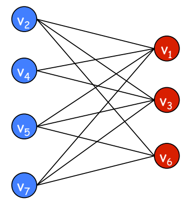
Testing Bipartiteness

Testing bipartiteness. Given a graph G, is it bipartite?

- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

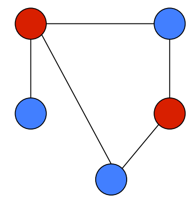


another drawing of G

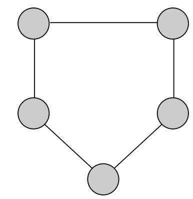
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

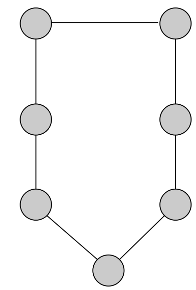
Pf. Impossible to 2-color the odd cycle, let alone G.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

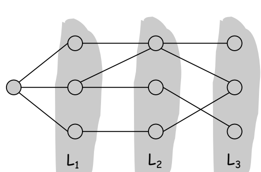


not bipartite
(not 2-colorable)

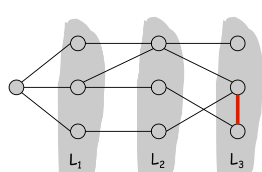
Bipartite Graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
- (ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)

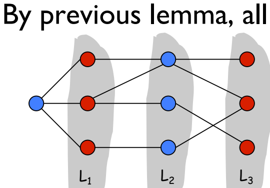
Bipartite Graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

- (i) No edge of G joins two nodes of the same layer, and G is bipartite.
- (ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

- Suppose no edge joins two nodes in the same layer.
- By previous lemma, all edges join nodes on adjacent levels.



Case (i)

Bipartition:
red = nodes on odd levels,
blue = nodes on even levels.

Bipartite Graphs

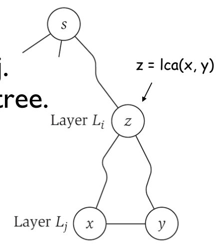
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

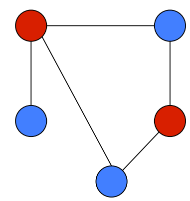
Suppose (x, y) is an edge & x, y in same level Lj.
Let z = their lowest common ancestor in BFS tree.
Let Li be level containing z.
Consider cycle that takes edge from x to y, then tree from y to z, then tree from z to x.
Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd.



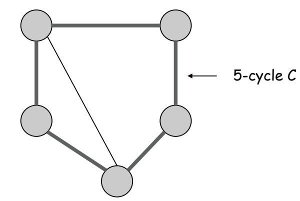
Obstruction to Bipartiteness

Cor: A graph G is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic-- in a non-bipartite graph, it *finds* an odd cycle.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

3.6 DAGs and Topological Ordering

Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

Applications

Course prerequisites: course v_i must be taken before v_j

Compilation: must compile module v_i before v_j

Job Workflow: output of job v_i is part of input to job v_j

Manufacturing or assembly: sand it before you paint it…

Spreadsheet evaluation: cell v_j depends on v_i

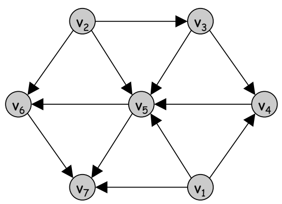
Directed Acyclic Graphs

Def. A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

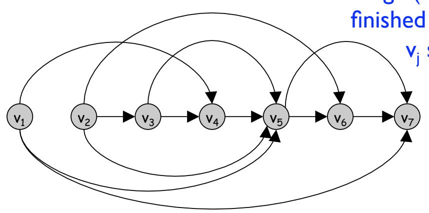
Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.

Def. A topological order of a directed graph G = (V, E) is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.

$\forall$ edge (v_i, v_j), v_i finished before v_j started



a DAG



a topological ordering of that DAG – all arrows go left-to-right

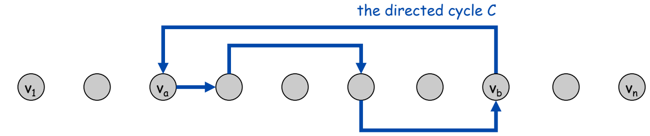
Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

if all edges go L→R, can't loop back to close a cycle

Pf. (by contradiction)

Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C.
Let v_a be the lowest-indexed node in C, and let v_b be the node just before v_a in the cycle; thus (v_b, v_a) is an edge.
By our choice of a, we have $a < b$.
On the other hand, since (v_b, v_a) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $b < a$, a contradiction. ▪



the supposed topological order: $v_1, \ldots, v_n$

Directed Acyclic Graphs

Lemma.
If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?

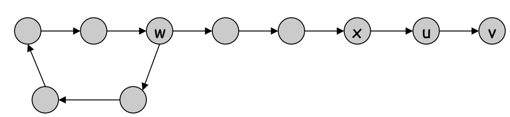
Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.
Then, since u has at least one incoming edge (x, u), we can walk backward to x.
Repeat until we visit a node, say w, twice.
Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.

Why must this happen?

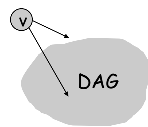


Directed Acyclic Graphs

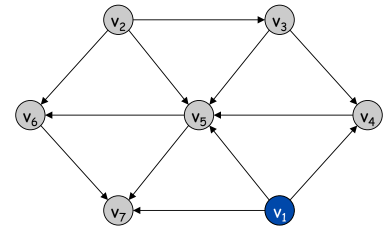
Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)
Base case: true if n = 1.
Given DAG on n > 1 nodes, find a node v with no incoming edges.
G - { v } is a DAG, since deleting v cannot create cycles.
By inductive hypothesis, G - { v } has a topological ordering.
Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges. ▪

To compute a topological ordering of G:
Find a node v with no incoming edges and order it first
Delete v from G
Recursively compute a topological ordering of $G-\{v\}$
  and append this order after v

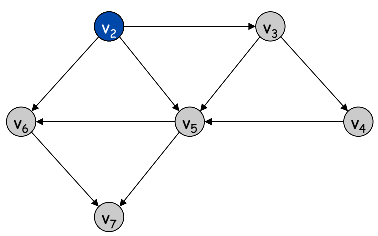


Topological Ordering Algorithm: Example



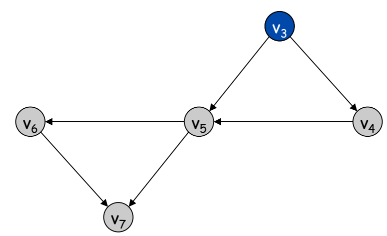
Topological order:

Topological Ordering Algorithm: Example



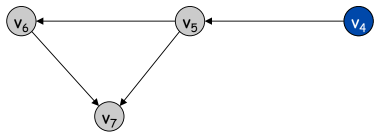
Topological order: v_1

Topological Ordering Algorithm: Example



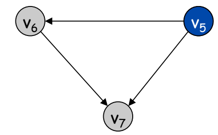
Topological order: v_1, v_2

Topological Ordering Algorithm: Example



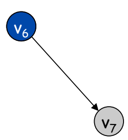
Topological order: v_1, v_2, v_3

Topological Ordering Algorithm: Example



Topological order: v_1, v_2, v_3, v_4

Topological Ordering Algorithm: Example

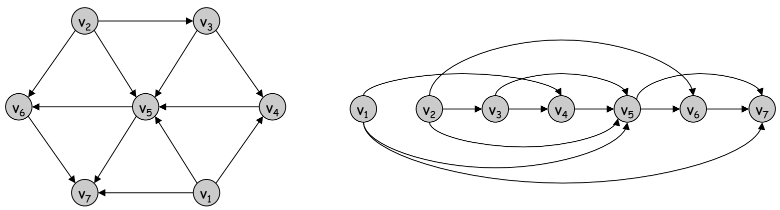


Topological order: v_1, v_2, v_3, v_4, v_5

Topological Ordering Algorithm: Example

Topological order: v_1, v_2, v_3, v_4, v_5, v_6

Topological Ordering Algorithm: Example



Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$.

Topological Sorting Algorithm

Maintain the following:

count[w] = (remaining) number of incoming edges to node w

S = set of (remaining) nodes with no incoming edges

Initialization:

count[w] = 0 for all w

count[w]++ for all edges (v,w)

S = S ∪ {w} for all w with count[w]==0

} O(m + n)

Main loop:

while S not empty

    remove some v from S

    make v next in topo order

    for all edges from v to some w

    decrement count[w]

    add w to S if count[w] hits 0

} O(1) per node, O(1) per edge

Correctness: clear, I hope

Time: O(m + n) (assuming edge-list representation of graph)

Depth-First Search

Follow the first path you find as far as you can go

Back up to last unexplored edge when you reach a dead end, then go as far you can

Naturally implemented using recursive calls or a stack

DFS(v) – Recursive version

Global Initialization:

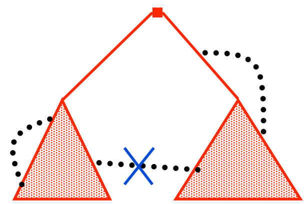
```
    for all nodes v, v.dfs# = -1   // mark v "undiscovered"
    dfscounter = 0

DFS(v)
    v.dfs# = dfscounter++         // v "discovered", number it
    for each edge (v,x)
        if (x.dfs# = -1)          // tree edge (x previously undiscovered)
            DFS(x)
        else …                    // code for back-, fwd-, parent,
                                  // edges, if needed
                                  // mark v "completed," if needed
```

Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

No cross edges!



Why fuss about trees (again)?

BFS tree ≠ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ancestor

Summary

Graphs –abstract relationships among pairs of objects

Terminology – node/vertex/vertices, edges, paths, multi-edges, self-loops, connected

Representation – edge list, adjacency matrix

Nodes vs Edges – m = $O(n^2)$, often less

BFS – Layers, queue, shortest paths, all edges go to same or adjacent layer

DFS – recursion/stack; all edges ancestor/descendant

Algorithms – connected components, bipartiteness, topological sort