CSE 417: Algorithms and Computational Complexity

Lecture 2: Analysis

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Defining Efficiency

"Runs fast on typical real problem instances"

Pro:

sensible, bottom-line-oriented

Con:

moving target (diff computers, compilers, Moore's law)

highly subjective (how fast is "fast"? what's "typical"?)

Efficiency

Our correct TSP algorithm was incredibly slow

Basically slow no matter what computer you have

We want a general theory of "efficiency" that is

- Simple
- Objective
- Relatively independent of changing technology
- But still predictive - "theoretically bad" algorithms should be bad in practice and vice versa (usually)

Measuring efficiency

Time ≈ # of instructions executed in a simple programming language

- only simple operations (+,*,-,=,if,call,…)
- each operation takes one time step
- each memory access takes one time step
- no fancy stuff (add these two matrices, copy this long string,…) built in; write it/charge for it as above

No fixed bound on the memory size

We left out things but...

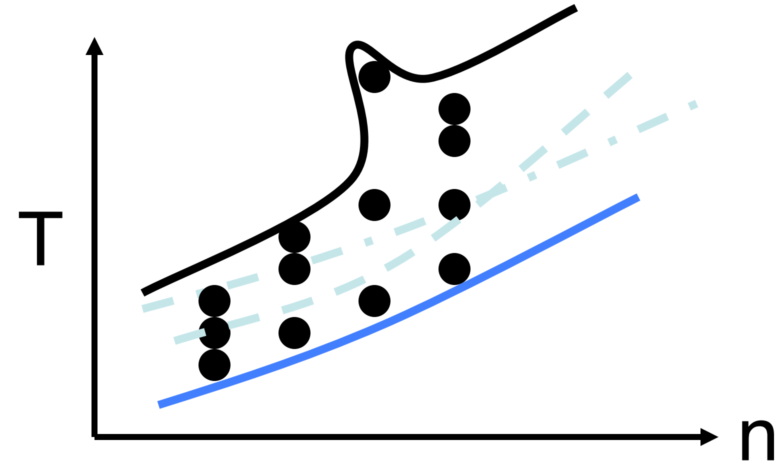
Things we’ve dropped

- memory hierarchy
  - disk, caches, registers have many orders of magnitude differences in access time
- not all instructions take the same time in practice
- different computers have different primitive instructions

However,

- the RAM model is useful for designing algorithms and measuring their efficiency
- one can usually tune implementations so that the hierarchy etc. is not a huge factor

Complexity analysis



Problem size n

- Worst-case complexity: max # steps algorithm takes on any input of size n
- Best-case complexity: min # steps algorithm takes on any input of size n
- Average-case complexity: avg # steps algorithm takes on inputs of size n

Pros and cons:

Best-case

- unrealistic oversell

Average-case

- over what probability distribution? (different people may have different "average" problems)
- analysis often hard

Worst-case

- a fast algorithm has a comforting guarantee
- maybe too pessimistic

Why Worst-Case Analysis?

Appropriate for time-critical applications, e.g. avionics

Unlike Average-Case, no debate about what the right definition is

- If worst >> average, then (a) alg is doing something pretty subtle, & (b) are hard instances really that rare?

Analysis often easier

Result is often representative of "typical" problem instances

Of course there are exceptions…

General Goals

Characterize growth rate of (worst-case) run time as a function of problem size, up to a constant factor

Why not try to be more precise?

- Technological variations (computer, compiler, OS, …) easily 10x or more
- Being more precise is a ton of work
- A key question is "scale up": if I can afford to do it today, how much longer will it take when my business problems are twice as large? (E.g. today: cn^2, next year: $c(2n)^2 = 4cn^2$: 4 x longer.)

Complexity

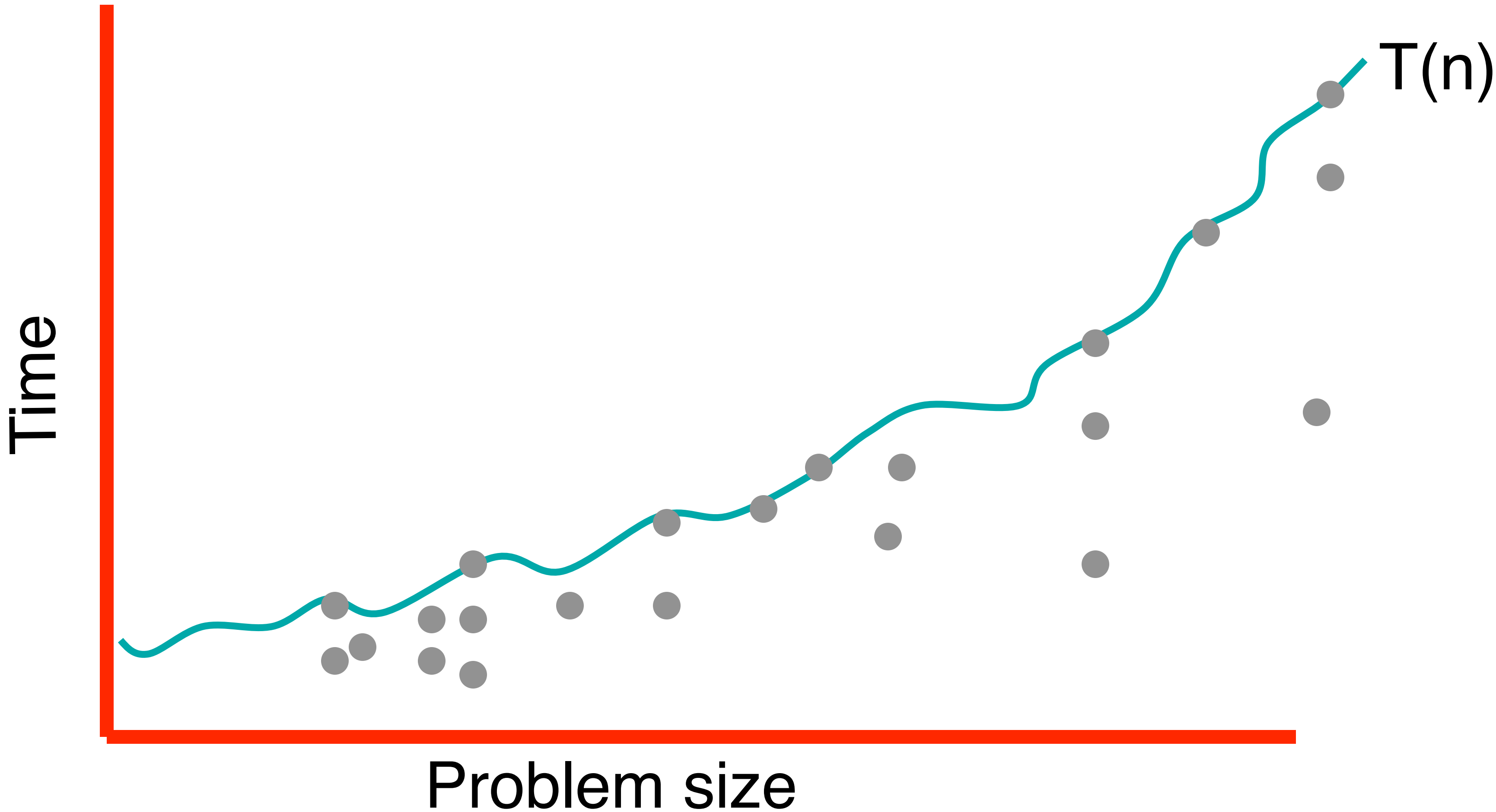
The complexity of an algorithm associates a number T(n), the worst-case time the algorithm takes, with each problem size n.

Mathematically,

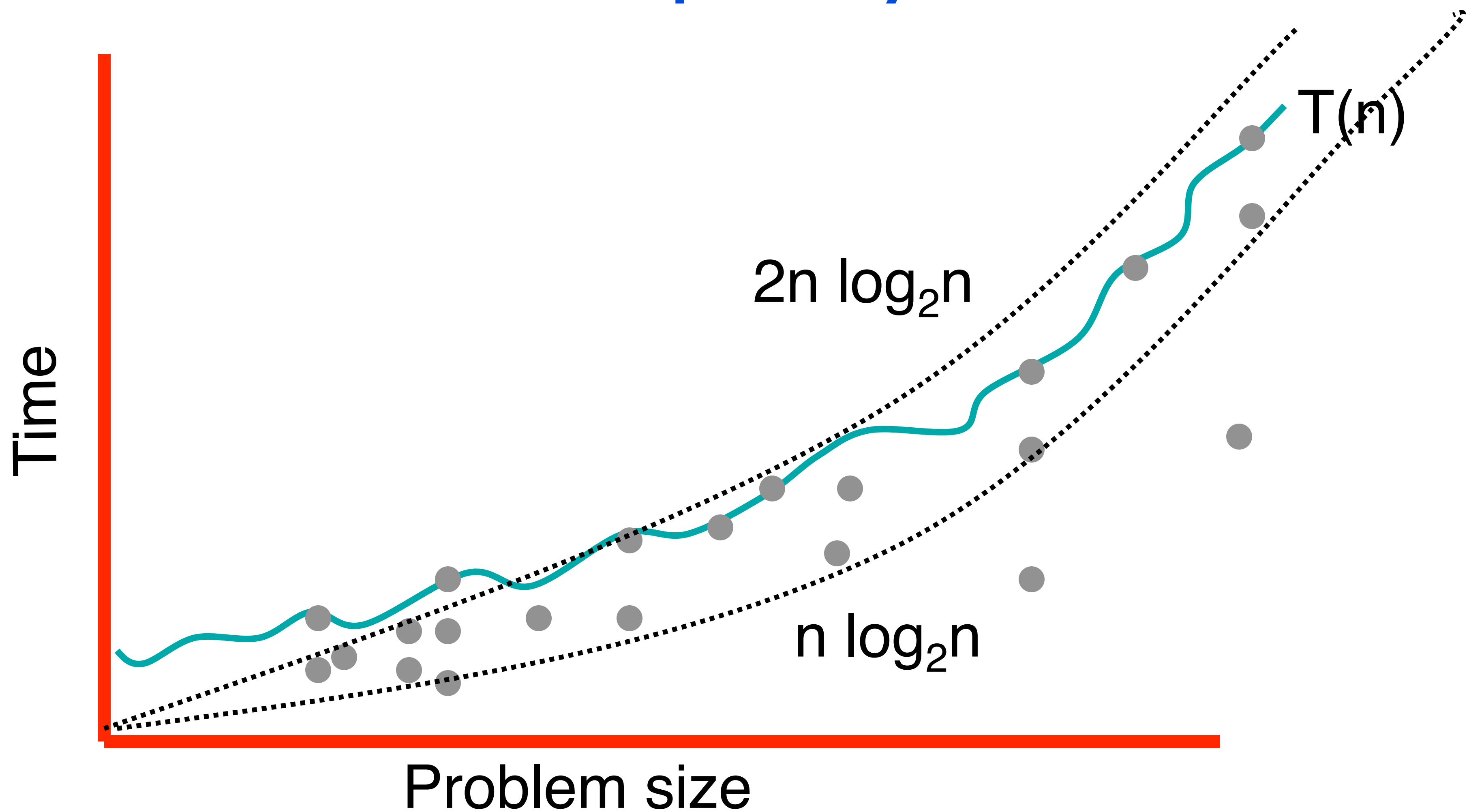
$T: N+ \rightarrow R+$

that is T is a function that maps positive integers (giving problem sizes) to positive real numbers (giving number of steps).

Complexity



Complexity



O-notation etc

Given two functions f and $g: N \rightarrow R$

f(n) is O(g(n)) iff there is a constant $c>0$ so that
f(n) is eventually always $\leq c\, g(n)$

f(n) is Ω (g(n)) iff there is a constant $c>0$ so that
f(n) is eventually always $\geq c\, g(n)$

f(n) is Θ (g(n)) iff there is are constants c_1, $c_2>0$ so that
eventually always $c_1 g(n) \leq f(n) \leq c_2 g(n)$

Examples

$10n^2-16n+100$ is $O(n^2)$ also $O(n^3)$

$10n^2-16n+100 \leq 11n^2$ for all $n \geq 10$

$10n^2-16n+100$ is $\Omega(n^2)$ also $\Omega(n)$

$10n^2-16n+100 \geq 9n^2$ for all $n \geq 16$

Therefore also $10n^2-16n+100$ is $\Theta(n^2)$

$10n^2-16n+100$ is not $O(n)$ also not $\Omega(n^3)$

Properties

Transitivity.

If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.

If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.

If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.

If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$.

Asymptotic Bounds for Some Common Functions

Polynomials:

$a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$

Logarithms:

$O(\log_a n) = O(\log_b n)$ for any constants $a,b > 0$

Logarithms:

For all $x > 0$, $\log n = O(n^x)$

log grows slower than every polynomial

"One-Way Equalities"

2 + 2 is 4

2 + 2 = 4

4 = 2 + 2

$2n^2 + 5\ n$ is $O(n^3)$

$2n^2 + 5\ n = O(n^3)$

$O(n^3) = 2n^2 + 5\ n$

All dogs are mammals

All mammals are dogs

Bottom line:

OK to put big-O in R.H.S. of equality, but not left.

[Better, but uncommon, notation: $T(n) \in O(f(n))$.]

Working with O-Ω-Θ notation

Claim: For any a, and any b>0, $(n+a)^b$ is $\Theta(n^b)$

$(n+a)^b \le (2n)^b$ for $n \ge |a|$

$= 2^b n^b$

$= cn^b$ for $c = 2^b$

so $(n+a)^b$ is $O(n^b)$

$(n+a)^b \ge (n/2)^b$ for $n \ge 2|a|$ (even if $a < 0$)

$= 2^{-b} n^b$

$= c'n$ for $c' = 2^{-b}$

so $(n+a)^b$ is $\Omega\ (n^b)$

Working with O-Ω-Θ notation

Claim: For any a, b>1 $\log_a n$ is Θ ($\log_b n$)

$$\log_a b = x \text{ means } a^x = b$$

$$a^{\log_a b} = b$$

$$(a^{\log_a b})^{\log_b n} = b^{\log_b n} = n$$

$$(\log_a b)(\log_b n) = \log_a n$$

$$c \log_b n = \log_a n \text{ for the constant } \mathrm{c} = \log_a b$$

$$\text{So}:$$

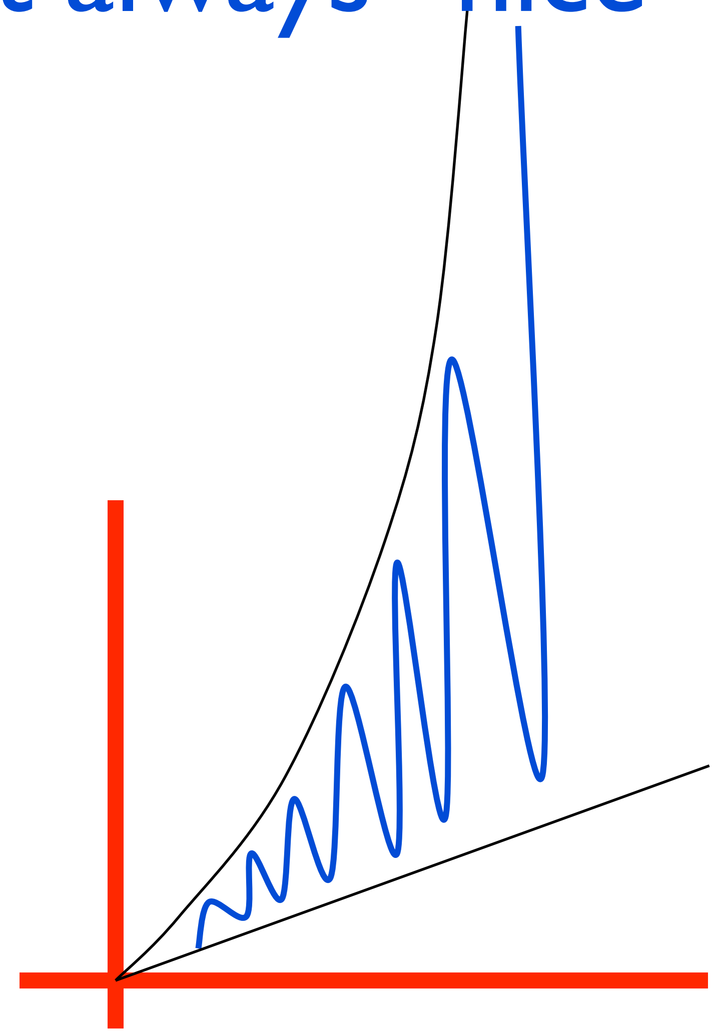
$$\log_b n = \Theta(\log_a n) = \Theta(\log n)$$

Big-Theta, etc. not always "nice"

$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$

f(n) $\neq \Theta(n^a)$ for any *a*.

Fortunately, such nasty cases are rare



f(n log n) $\neq \Theta(n^a)$ for any *a, either, but at least it's simpler*.

A Possible Misunderstanding?

We have looked at

- type of complexity analysis
  - worst-, best-, average-case
- types of function bounds
  - O, Ω, Θ

These two considerations are independent of each other

- one can do any type of function bound with any type of complexity analysis - measuring different things with same yardstick

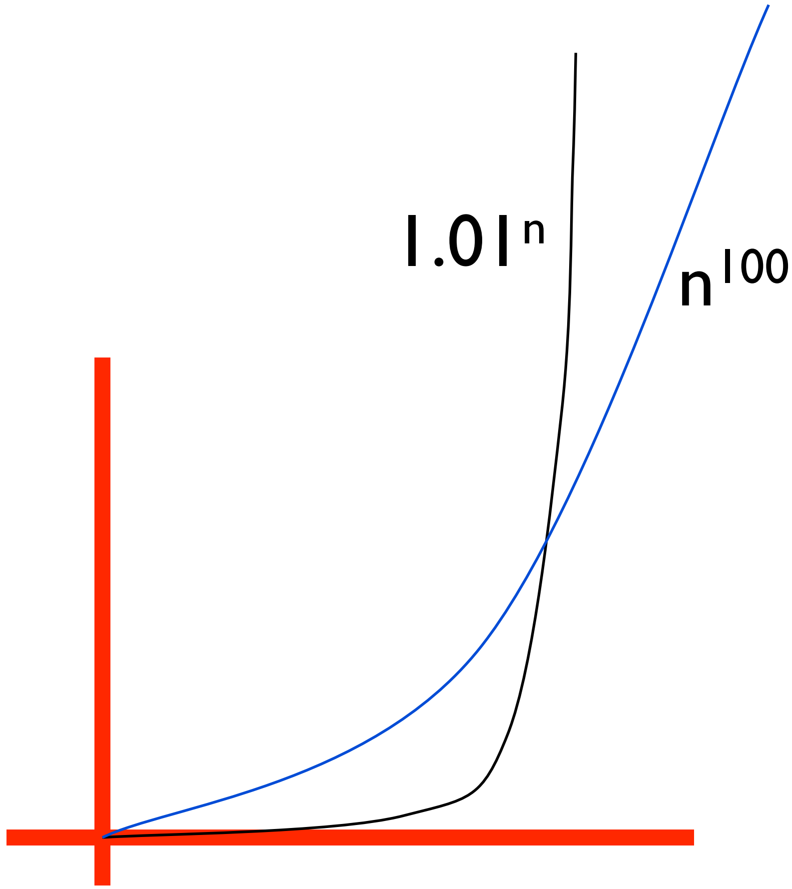
Insertion Sort:

- $\Omega(n^2)$ (worst case)
- $O(n)$ (best case)

Asymptotic Bounds for Some Common Functions

Exponentials.
For all $r > 1$
and all $d > 0$,
$n^d = O(r^n)$.

every exponential
grows faster than
every polynomial



Polynomial time

Running time is $O(n^d)$ for some constant d independent of the input size n.

Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	4 sec
$n = 30$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	18 min	10^{25} years
$n = 50$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	11 min	36 years	very long
$n = 100$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	$<$ 1 sec	$<$ 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	$<$ 1 sec	$<$ 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	$<$ 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Geek-speak Faux Pas du Jour

"Any comparison-based sorting algorithm requires at least $O(n \log n)$ comparisons."

- Statement doesn't "type-check."
- Use Ω for lower bounds.

Summary

Typical initial goal for algorithm analysis is to find a

- reasonably tight ← i.e., Θ if possible
- *asymptotic* ← i.e., O or Θ
- bound on ← usually upper bound
- *worst case* running time
- as a function of problem *size*

This is rarely the last word, but often helps separate good algorithms from blatantly poor ones - so you can concentrate on the good ones!